Hopf Fibration and Spinors

Simão Santos

January 2024

1 Introduction

As the topic intersects a multitude of fields we deem wise to present the reader some useful concepts that will help clarify the connection between the Hopf Fibration and Spinors. We start with some definitions and then proceed to the subject itself, ending with a simple case of a Spinor system, the *spin* $-$ 1/2 *system*. Other important concepts necessary to understand this paper, such as the concepts of manifold and vector bundle, are referenced for the less experienced reader. (See [4] for instance)

Definition 1 *A fiber bundle is a tuple* $(\mathcal{E}, \mathcal{B}, F, p)$*, where* $\mathcal{E}, \mathcal{B}$ F *are topological spaces and* $p : \mathcal{E} \to \mathcal{B}$ *is a map with the following properties:*

1. *For every point* $z \in \mathcal{B}$ $\exists\, \mathcal{U}$*, an open set of* $\mathcal{B}$ *whose pre-image* $p^{-1}(\mathcal{U})$ *is homeomorphic to* $\mathcal{U} \times F$
2. *The homeomorphism* $\psi : \mathcal{U} \times F \to p^{-1}(\mathcal{U})$ *makes the following diagram commute:*

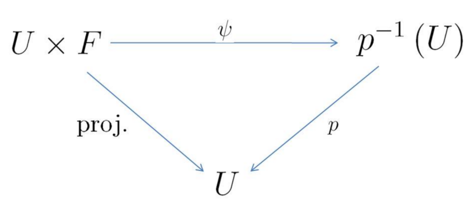

Figure 1: Fiber Bundle Diagram

Definition 2 *In a fiber bundle as above,* $\mathcal{E}$ *is a fiber space,* $\mathcal{B}$ *is the base space,* F *is the fiber, and* p *is the projection.*

2 Hopf Fibration

This structure appears as one of the first and most influential examples of a non-trivial fiber bundle, despite being locally indistinguishable from the product of $S^2 \times S^1$. It is part of a family of four fiber bundles

$$S^0 \hookrightarrow S^1 \to S^1$$
$$S^1 \hookrightarrow S^3 \to S^2$$
$$S^3 \hookrightarrow S^7 \to S^4$$
$$S^7 \hookrightarrow S^{15} \to S^8$$

and it is known that such fibrations can only occur in these dimensions (Adam's Theorem, see [1]).

The Hopf Fibration defines a map $p : S^3 \longrightarrow S^2$ with fibers S^1. One way to see S^1 as a fiber of S^3 is to consider the action of S^1 on S^3, which is a proper and free action so that S^3/S^1 is a smooth manifold and $\pi : S^3 \longrightarrow S^3/S^1$ is a submersion. It is possible to show that there is a diffeomorphism between S^3/S^1 and S^2, thus inducing the Hopf map, p. Our goal here is not to prove what is claimed above but to highlight the connection between the Hopf Fibration and its applications in Physics, particularly in Spinors. With this in mind, we now present some known properties of the Hopf Fibration. Firstly, consider the following definition:

Definition 3 *The equivalence relation* $\sim$ *on the set* $\mathbb{C}^2$ *is defined by* $(z_1, z_2) \sim (w_1, w_2)$ *if* $\exists c \in \mathbb{C} \setminus \{0\}$ *such that* $(z_1, z_2) = c \cdot (w_1, w_2)$*.*

Now note that:

$$S^3 = \{x, y, z, t \in \mathbb{R}|\ x^2 + y^2 + z^2 + t^2 = 1\} \Leftrightarrow$$
$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2|\ z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$$

The more attentive reader can find this interesting since, when applying the equivalence relation to S^3, one is simply describing the action of S^1 on S^3, i.e.:

Proposition 1 *The equivalence classes defined by the relationship above are generated by the orbit of the action of* S^1 *on points of* S^3*, with the difference that now that the factor relating to elements of the same class,* λ*, follows the condition imposed by the definition of* S^3*:*

For any (w_1, w_2) , (z_1, z_2) *such that* $\exists \lambda \in \mathbb{C}$ *s.t.* $(w_1, w_2) = \lambda \cdot (z_1, z_2)$ *then*

$$1 = w_1\bar{w_1} + w_2\bar{w_2} = \lambda\bar{\lambda}(z_1\bar{z_1} + z_2\bar{z_2}) = \lambda\bar{\lambda}$$

Thus implying that $|\lambda| = 1$*.*

Defining the Hopf map $p : S^3 \longrightarrow S^2$ by $p(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$, the pre-image of $p(z_1, z_2)$ is then the set of points obtained from (z_1, z_2) by multiplying each coordinate by $e^{i\theta}$, where $-\pi < \theta \leq \pi$. From this, the drawn conclusion is that each point in S^2 is a great circle in S^3. Therefore, $\forall s \in S^2, p^{-1}(s)$ is homeomorphic to S^1. This justifies the definition of the action of S^1 in S^3.

Consider $\mathbb{CP}^1$ (the complex projective plane) the set of equivalence classes of $\mathbb{C}^2 \setminus \{0\}$, with the equivalence classes defined by means of homogeneous coordinates $[z_1 : z_2]$. We then have the correspondence $\mathbb{CP}^1 = \{[z : 1]|\ z \in \mathbb{C}\} \cup [1 : 0]$, from which it can be shown that $\mathbb{CP}^1$ is homeomorphic to $\mathbb{C} \cup \infty$ (See [3] for more details).

With the above construction, the Hopf map can now be redefined as the following $p : S^3 \longrightarrow \mathbb{CP}^1$ where $p : (z_1, z_2) = [z_1 : z_2]$, since $\mathbb{CP}^1$ is homeomorphic to S^2. From this we get the map f induced by p, $f : S^3 \longrightarrow \mathbb{C} \cup \{\infty\}$, mapping $(z_1, z_2) \in S^3$ to the point $z_1/z_2 \in \mathbb{C} \cup \{\infty\}$. This construction will be important in no time since we will be employing it in Section 3.

3 Physical application: Spinors

The Hopf fibration turns out to be of the utmost importance in Physics, particularly in Quantum Mechanics. In this theory, physical systems are described by a mathematical object known as a state. Contrarily to what happens in Classical Mechanics, the so-called states are not bundles of specific values for different properties of the system, but rather hold the various probabilities of observing all the possible physically measurable quantities.

One of the easiest and most practical examples to work with in Quantum Mechanics is the *spin* $1/2 - system$, as in the case of the electron, which has spin-up (ψ_+) and spin-down(ψ_-). These two configurations are exclusive outcomes of a measurement, occurring with probability p_+ and p_-, respectively. These two configurations are orthogonal and form an orthonormal basis of the state space.

In order to have a consistent description of the system, if two electrons have the same values for p_+ and p_-, they are said to be in the same quantum state, despite the fact they can be in different mixtures of p_+ and p_-. This gives us a hint on how to proceed in the formalization of the quantum state: the quantum state (a theoretical mathematical object) and the physical reality must coincide. The quantum state of an electron must then be described by an equivalence class over a set of states that yield the same physical result.

Definition 4 *$\mathbb{C}^2$ can be equipped with the inner product operator $\langle\cdot|\cdot\rangle : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$, which is defined by the following properties:*

1. *Symmetry: $\forall \psi, \phi \in \mathbb{C}^2, \langle\psi|\phi\rangle = \overline{\langle\psi|\phi\rangle}$*
2. *Bilinearity: $\forall \psi, \phi, \zeta \in \mathbb{C}^2$ and $a, b \in \mathbb{C}, \langle a\psi + b\phi|\zeta\rangle = \bar{a}\langle\psi|\zeta\rangle + \bar{b}\langle\phi|\zeta\rangle$ and $\langle\psi|a\phi + b\zeta\rangle = a\langle\psi|\zeta\rangle + b\langle\phi|\zeta\rangle$*
3. *Positive definiteness: $\forall \psi \in \mathbb{C}^2, \langle\psi|\psi\rangle \in \mathbb{R} \subseteq \mathbb{C}$ and $\langle\psi|\psi\rangle \geq 0$ and $\langle\psi|\psi\rangle = 0$ if and only if $\psi = 0$.*

$\mathbb{C}^2$ is an example of a Hilbert Space.

It follows that the inner product of two state vectors $\psi = (\psi_1, \psi_2)$ and $\phi = (\phi_1, \phi_2)$ can be computed by

$$\langle\psi|\phi\rangle = \overline{\psi_1}\phi_1 + \overline{\psi_2}\phi_2.$$

Definition 5 *The norm of the inner product is given by the map $||\cdot|| : \mathbb{C}^2 \longrightarrow \mathbb{R}$ such that:*

$$||\psi|| = \sqrt{\langle\psi|\psi\rangle}, \forall\psi \in \mathbb{C}^2.$$

Definition 6 *The state of an electron is defined as an element $\psi \in S(\mathbb{C}^2)$, where $S(\mathbb{C}^2) = \{\psi \in \mathbb{C}^2 : ||\psi|| = 1\}$.*

Definition 7 *Two states $\psi, \phi \in S(\mathbb{C}^2)$ are considered equivalent if $\exists\lambda \in \mathbb{C} \setminus \{0\}$ such that $\phi = \lambda\psi$.*

Note that for the definitions above to be physically consistent, any two electrons in the same quantum state must, upon measurement, return similar outcomes. The measurement of an observable of the system can return many outcomes. What we are really looking for is for the expected value of these measurements to coincide.

In Quantum Mechanics, the expected value of an observable is determined in the following way:

Proposition 2 *Each observable is associated with a linear operator $\hat{O}$. The expected value of the observable $\hat{O}$ over a system in quantum state ψ, $\overline{O}_\psi$, is given by*

$$\overline{O}_\psi = \langle\psi|\hat{O}|\psi\rangle$$

For the *spin - 1/2 system* the observables are the 3 - dimensional components of the spin vector $s = (s_x, s_y, s_z)$. Each of these components is an observable whose related linear operators are the Pauli-Matrices:

$$\hat{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 3 *Two electrons in the same quantum state $[\psi]$ have the same expected value for the spin vector $\overline{s} = \overline{s}_x\hat{x} + \overline{s}_y\hat{y} + \overline{s}_z\hat{z}$. This implies that there exists a one-to-one correspondence between quantum states and expected values of the spin vector, which is guaranteed by the Hopf Fibration.*

To see why this holds, we work over $S(\mathbb{C}^2)$, which is homeomorphic to S^3. Note that there is a correspondence between spin vectors and points of S^2.

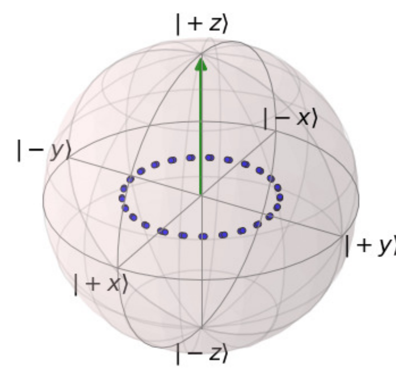

Figure 2: Spin Visualization

Consider the map $g : S^3 \longrightarrow S^2$, which takes $\psi \mapsto \bar{s}(\psi)$. Then:

$$g\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \langle\psi|\hat{S}_x\psi\rangle\hat{x} + \langle\psi|\hat{S}_y\psi\rangle\hat{y} + \langle\psi|\hat{S}_z\psi\rangle\hat{z} =$$

$$\left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} \right\rangle \hat{x} + \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{pmatrix} -i\psi_2 \\ i\psi_1 \end{pmatrix} \right\rangle \hat{y} + \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{pmatrix} -\psi_1 \\ \psi_2 \end{pmatrix} \right\rangle \hat{z}$$

$$= (\overline{\psi_1}\psi_2 + \overline{\psi_2}\psi_1)\hat{x} + i(\overline{\psi_2}\psi_1 - \overline{\psi_1}\psi_2)\hat{y} + (\overline{\psi_1}\psi_1 - \overline{\psi_2}\psi_2)\hat{z}$$

Once again we now have a hint on how to proceed, let $z = x + iy$, then:

$$x = \frac{\overline{\psi_1}\psi_2 + \overline{\psi_2}\psi_1}{1 - \overline{\psi_1}\psi_1 + \overline{\psi_2}\psi_2} = \frac{\overline{\psi_1}\psi_2 + \overline{\psi_2}\psi_1}{2\overline{\psi_2}\psi_2} = Re\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$y = -\frac{i(\overline{\psi_2}\psi_1 - \overline{\psi_1}\psi_2)}{1 - \overline{\psi_1}\psi_1 + \overline{\psi_2}\psi_2} = -\frac{i(\overline{\psi_2}\psi_1 - \overline{\psi_1}\psi_2)}{2\overline{\psi_2}\psi_2} = Im\left(\frac{\psi_1}{\psi_2}\right)$$

If through the stereographic projection we view S^2 as the extended complex plane, then the map $g : (\psi_1, \psi_2) \longrightarrow \psi_1/\psi_2$ is precisely the quotient map $f : S^3 \longrightarrow S^2$ defined before. This map f is the exact same as the Hopf map p redefined for $\mathbb{CP}^1$, since $\mathbb{CP}^1$ is the quotient space of S^3 with $(z_1, z_2) \sim (w_1, w_2)$ iff $(z_1, z_2) = c(w_1, w_2)$ for some $c \in \mathbb{C}$, which is true iff $z_1/z_2 = w_1/w_2$. Hence there is a correspondence between ψ_1/ψ_2 and $[\psi_1 : \psi_2]$ and thus the map $g : \psi \longrightarrow \bar{s}(\psi)$ is the fiber map of the Hopf Fibration $S^1 \hookrightarrow S^3 \to S^2$.

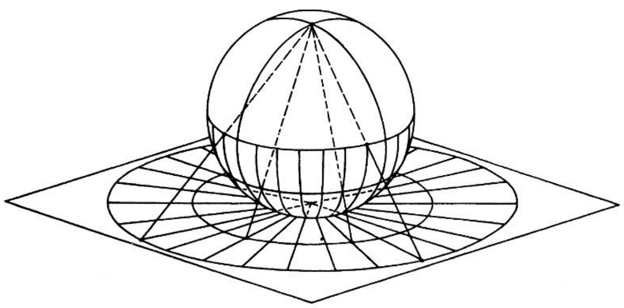

Figure 3: Stereographic Projection

Note that g sends $\psi, \phi \in S^3$ to the same image iff $\psi = \lambda\phi$, where $|\lambda| = 1$. This tells us that they are in the same equivalence class of the Hopf map, i.e., they represent the same quantum state, which establishes the so desired correspondence we were looking for.

4 Conclusion

This small paper sheds some light on the connection between Hopf Fibration and Spinors, as exemplified by the case of the *spin* $-1/2$ *system*. This structure is what motivates the consistency between the mathematical theory of Quantum Mechanics behind Spinors and the physical reality observed in the corresponding systems. We covered a simple case, but more can be found about this relationship (See [2] for instance).

5 Bibliography

1. https://en.wikipedia.org/wiki/Hopf_fibration
2. https://arxiv.org/abs/1601.02569
3. https://nilesjohnson.net/hopf-articles/Hongwan_Liu-Hopf_fibration.pdf
4. "Differential Geometry" by Rui Loja Fernandes (available at https://nmath.tecnico.ulisboa.pt/repositorio/browser.php?path=.%2Fdocs%2FPrograma+Doutoral+em+Matem%C3%A1tica%2F%5BGDif%5D+Geometria+Diferencial%2FBibliografia%2F)