Frieze Groups

Instituto Superior Técnico
Mestrado em Matemática e Aplicações
Métodos de Álgebra e Geometria para Engenharia e Física

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1 Isometries

The first step in studying Frieze Groups is understanding what an Isometry is. Given an Euclidian Space $\mathcal{E}$, a map $f : \mathcal{E} \to \mathcal{E}$ is called an *Isometry* if $\forall P, Q \in \mathcal{E}$ we have:

$$d(P,Q) = d(f(P), f(Q)) \tag{1}$$

where $d(P,Q)$ is the *Euclidean Distance* [1].
The composition $f \circ g$ of two Isometries f and g is also an Isometry.
There are four types of Plane Isometries:

- A *Translation*, that is a map $T_v : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_v(P) = P + v$, where P is a point of $\mathbb{R}^2$ and v a vector of $\mathbb{R}^2$.

- A *Rotation of centre C and angle θ*, given by the map $\rho_{C,\theta} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\rho_{C,\theta}(P) = C + \overrightarrow{\Phi_\theta}(\overrightarrow{CP}), \forall P \in \mathbb{R}^2 \tag{2}$$

  where the orthogonal map $\overrightarrow{\Phi_\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation of center $(0,0)$ and angle θ and has associated the following matrix:

$$M_{\overrightarrow{\Phi_\theta}} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \tag{3}$$

- A *Reflection on a line l*, given by the map $R_l : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$R_l(P) = \begin{cases} P & P \in l \\ P' & P \notin l \end{cases} \tag{4}$$

  where $P, P' \in \mathbb{R}^2$ and P' is the symmetric of P with respect to l.

- A *Gliding Reflection* is the composition of a reflection and a translation: $T_v \circ R_l$.

We are now all set to start discussing Frieze Groups.

2 Frieze Groups

A *frieze* is an image that repeats itself endlessly in the same direction.
The essential property of a frieze pattern is that it is left fixed by some "smallest translation", although there are other isometries present in a frieze pattern.
A group of isometries that fix a line l and whose translation τ form an infinite cyclic group is a *frieze group* with *center* l. There are seven different frieze groups that can be defined through their isometries.

Let $\mathcal{F}$ be a frieze group with center l whose translations form the group generated by translation τ and let σ_l be the group of reflections in the line l. If $\mathcal{F}$ contains a halfturn (rotation of 180º), suppose $\mathcal{F}$ contains σ_A (the group of halfturns with center A). If $\mathcal{F}$ has a reflection in a line a perpendicular to l, suppose $\mathcal{F}$ contains σ_a (the group of reflections in the line a). Finally, let γ be the glide reflection such that $\gamma = \tau^2$ [3]. There are seven frieze groups, and here we have a brief description, such as an illustrative example of them:

- $\mathcal{F}_1 = \langle \tau \rangle$, formed only by translations, having no lines or points of symmetry 1.
- $\mathcal{F}_1^1 = \langle \tau, \sigma_l \rangle$, formed by glide reflections, translations and reflections with axis l and having no point of symmetry and l as the line of symmetry 2.
- $\mathcal{F}_1^2 = \langle \tau, \sigma_a \rangle$, formed by translations and reflections with axis a, having no point of symmetry and a as the line of symmetry 3.
- $\mathcal{F}_1^3 = \langle \gamma \rangle$, formed by translations and glide reflections, having no lines or points of symmetry 4.
- $\mathcal{F}_2 = \langle \tau, \sigma_A \rangle$, contains halfturns and translations, having a point of symmetry but no line of symmetry 5.
- $\mathcal{F}_2^1 = \langle \tau, \sigma_A, \sigma_l \rangle$ contains translations, halfturns, reflection with axis a and l and glide reflections, having a line of symmetry but no point of symmetry 6.
- $\mathcal{F}_2^2 = \langle \gamma, \sigma_A \rangle$ contains translations, halfturns, reflection with axis a and glide reflections, having a line of symmetry and a point of symmetry 7.

The easiest way to check that frieze groups are, indeed, groups, is by classifying each frieze pattern into an isomorphism class.
Firstly, we can see that $\mathcal{F}_1$ and $\mathcal{F}_1^3$ are isomorphic to $\mathbb{Z}$, since they can be described solely by translations, being singly generated. $\mathcal{F}_1^2$, $\mathcal{F}_2$ and $\mathcal{F}_2^2$ will be isomorphic to $\mathbb{D}_\infty$, because they are defined through translations (isomorphic to $\mathbb{Z}$) and reflections or rotations (hence not being abelian), and are doubly generated.
$\mathcal{F}_1^1$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$, being the only group with two generators that is abelian. Finally, $\mathcal{F}_2^1$ is isomorphic to $\mathbb{D}_\infty \oplus \mathbb{Z}_2$, having three generators [2].

3 Friezes in Architecture

The definition of a frieze in architecture is not the same as in mathematics. In architecture a frieze is defined as the wide central section of an entablature, or, more loosely, as any continuous horizontal strip of decoration. The concept of a frieze in mathematics comes from the "architectural meaning".

Although we can find examples of friezes in various Architectural Styles, they are highly associated to Classical Architecture. We can see this in buildings like Erechtheion, an ancient Greek Ionic Temple 9.
A great use of frieze pattern in Architecture is the Romanesque church of Saint Peter of Abragão, in Penafiel 8.
The most famous frieze in architecture is probably the Parthenon's, however, it is not a frieze pattern, since it is composed by different figures, and therefore having no symmetry 10.

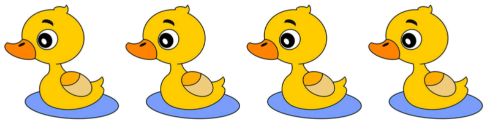

Figure 1: *Illustration* $\mathcal{F}_1$

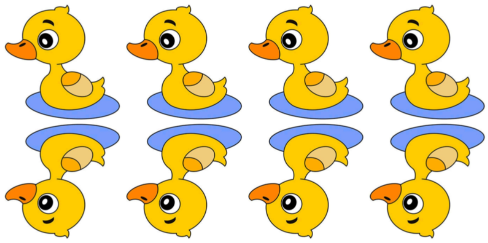

Figure 2: *Illustration* $\mathcal{F}_1^1$

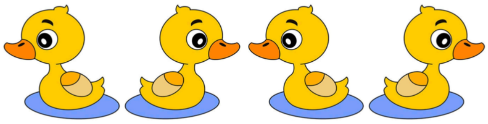

Figure 3: *Illustration* $\mathcal{F}_1^2$

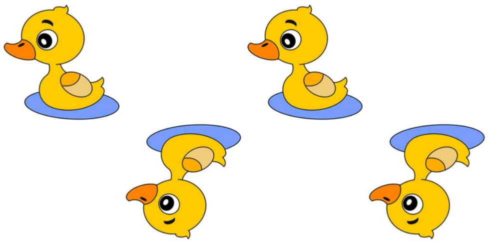

Figure 4: *Illustration* $\mathcal{F}_1^3$

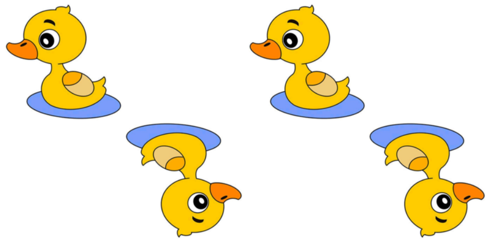

Figure 5: *Illustration* $\mathcal{F}_2$

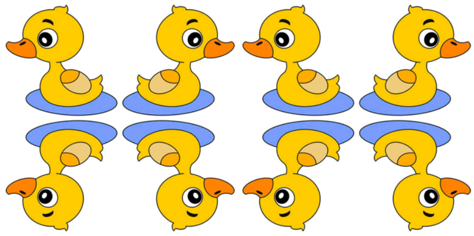

Figure 6: *Illustration* $\mathcal{F}_2^1$

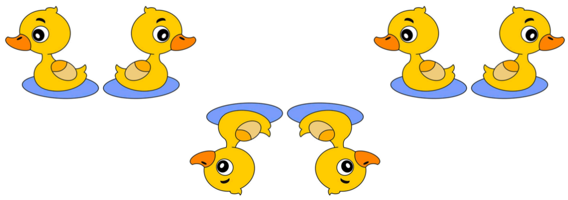

Figure 7: *Illustration* $\mathcal{F}_2^2$

Figure 8: *Example of a Frieze in a Portuguese Church*

Figure 9: *Example of Friezes in a Greek Temple*

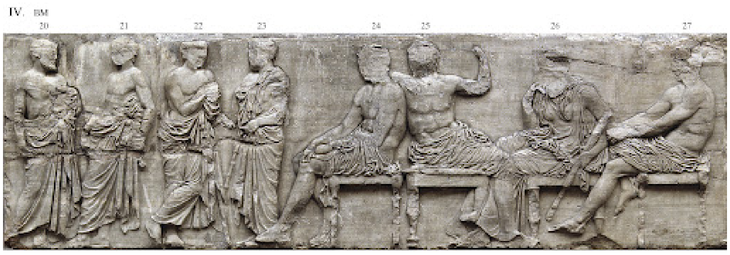

Figure 10: *Example of a (not) Frieze*

References

[1] Godinho, L. *Geometry Lecture Notes - Geometria Afim*.

[2] Landau, T. *Classifications of Frieze Groups and an Introduction to Crystallographic Groups*.

[3] MArtin, G. E. *Transformation Geometry - An Introduction to Symmetry*. 1987.