INSTITUTO SUPERIOR TÉCNICO

ALGEBRAIC AND GEOMETRIC METHODS IN ENGINEERING AND PHYSICS

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Frieze Groups

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1 Introduction

Symmetries surround us in our daily lives, evident in the lines of buildings, patterns on sidewalks, decorative elements in pottery and rugs, and even in nature.

Figure 1: Examples of symmetries in the sidewalk, in pottery and in Greek architecture

In Greek architecture, the 'frieze' refers to the section of a structure between the support beams and the top of the structure. However, in general, the term 'frieze' is commonly used to describe any continuous horizontal strip of decoration, containing figurative or ornamental motifs.

In mathematics, a frieze pattern is defined as a design on a two-dimensional surface that repeats in one direction. These patterns can be classified into seven different types based on their symmetries, known as the frieze groups.

2 Isometries of a Frieze Pattern

A symmetry group is a group specific to an object, where its elements leave the object invariant. These element are isometries - transformations that preserve the distance relationships within the object.

A frieze group represents the set of symmetries of a frieze pattern under composition. We will see in this report why there are exactly seven frieze groups, but first let's examine the isometries that compose these groups.

In the symmetry group of frieze patterns, we find six isometries: the identity (e), translations (t), horizontal reflections (h), vertical reflections (v), rotations (r), and glide reflections (g). These will be represented by the corresponding letters. A translation is a linear shift of some figure (a motif) along the plane. The rotation is a rotation of 180 degrees. A glide reflections is a combination of a translation and a horizontal reflection.

We illustrate how these multiple isometries act on Figure (2).

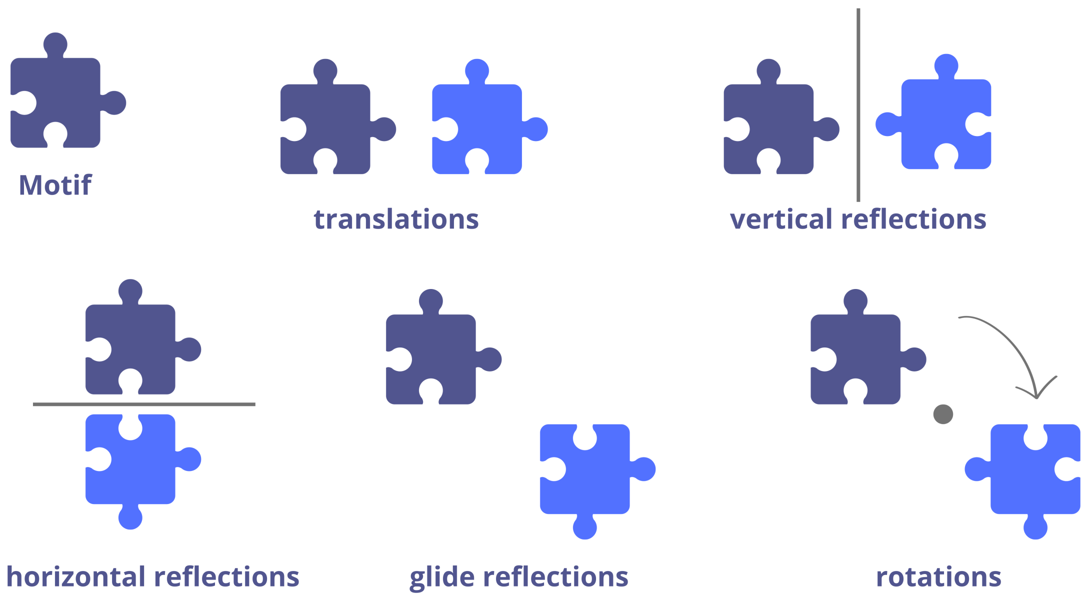

Figure 2: Isometries of frieze patterns

Now, let's explore some properties of the compositions of these isometries.

Firstly, we observe that a glide reflection can be expressed as either a translation followed by a horizontal reflection or a horizontal reflection followed by a translation. So, g is a composition of t and h. This also implies that horizontal reflection (h) and translation (t) commute, and $g = ht = th$. Additionally, a glide reflection commutes with a translation, leading to $gt = tg$.

Furthermore, we know that $h^2 = e$ and $v^2 = e$. However, unlike horizontal reflection (h), the vertical reflection (v) does not commute with a translation. Specifically, $vt = t^{-1}v$.

Lastly, a vertical reflection can be represented as a rotation followed by a horizontal reflection, denoted as $v = hr$. Similar to vertical reflections, rotations follow the rule $rt = t^{-1}r$ and $r^2 = e$.

With this foundation, we can now delve into the analysis of the symmetry groups.

3 The 7 Frieze Groups

All possible symmetries groups are generated by one or more isometries. But, we need to take in consideration the fact that all frieze groups inherently contain the translation (t) since they are defined as patterns that repeat in one direction. So we can exclude right away the groups only generated by combinations of h, v and r once the composition of this isometries will never lead to a translation. Our first two groups will be:

Group 1 - Has t as the generator: $\langle t \rangle$
Group 2 - Has g as the generator: $\left\langle g \mid g^2 = t^2 \right\rangle$

t^2 is also a translation so group 2 follows the definition. Now we look at all other possible groups with combinations of our isometries, making sure that they contain translations: $\langle t, h \rangle$, $\langle t, v \rangle$, $\langle t, r \rangle$, $\langle t, g \rangle$, $\langle t, h, v \rangle$, $\langle t, h, r \rangle$, $\langle t, h, g \rangle$, $\langle t, v, r \rangle$, $\langle t, v, g \rangle$, $\langle t, r, g \rangle$, $\langle t, h, v, r \rangle$, $\langle t, h, v, g \rangle$, $\langle t, h, g, r \rangle$ and $\langle t, v, g, r \rangle$, $\langle t, h, v, r, g \rangle$

This is quite a extensive list. Using the previously mentioned relations, let's analyze the remaining possible groups. We already know that g is a composition of h and t, $g = ht$, and that all groups must be closed under composition. So we can exclude right away all groups that have h, t and g has generators :

- $\langle t, h, g \rangle = \langle t, h, ht \rangle = \langle t, h \rangle$
- $\langle t, h, v, g \rangle = \langle t, h, v, ht \rangle = \langle t, h, v \rangle$
- $\langle t, h, g, r \rangle = \langle t, h, ht, r \rangle = \langle t, h, r \rangle$
- $\langle t, h, v, r, g \rangle = \langle t, h, v, r, ht \rangle = \langle t, h, v, r \rangle$

Using that $v = hr$, we have $\langle t, h, v, r \rangle = \langle t, h, hr, r \rangle = \langle t, h, r \rangle$.

Let's show that $\langle t, h, v \rangle = \langle t, r, h \rangle$: We start by seeing that $\langle t, h, v \rangle = \langle t, h, hr \rangle \subseteq \langle t, h, r \rangle$. Then we can write r as a combination of hr and h, $r = h^{-1}hr$ which means that $r \in \langle t, h, hr \rangle$ and that $\langle t, h, r \rangle \subseteq \langle t, h, hr \rangle$.

Following the same reasoning we will have that

- $\langle t, h, v \rangle = \langle t, h, hr \rangle = \langle t, h, r \rangle$
- $\langle t, v, r, g \rangle = \langle t, hr, r, g \rangle = \langle t, h, r, g \rangle = \langle t, h, r \rangle$
- $\langle t, g \rangle = \langle t, th \rangle = \langle t, h \rangle$

- $\langle t, r, g\rangle = \langle t, r, ht\rangle = \langle t, r, h\rangle$
- $\langle t, v, g\rangle = \langle t, v, ht\rangle = \langle t, v, h\rangle = \langle t, hr, h\rangle = \langle t, r, h\rangle$

This leaves us with 5 groups: $\langle t, h\rangle$, $\langle t, v\rangle$, $\langle t, r\rangle$, $\langle t, v, r\rangle$, $\langle t, h, v\rangle$

Group 3 - Has t and h as generators: $\left\langle t, h \mid h^2 = e; th = ht\right\rangle$
Group 4 - Has t and v as generators: $\left\langle t, v \mid v^2 = e; tv = vt^{-1}\right\rangle$
Group 5 - Has t and r as generators: $\left\langle t, r \mid r^2 = e; tr = rt^{-1}\right\rangle$
Group 6 - Has t,v and r as generators: $\left\langle t, v, r \mid v^2 = e; r^2 = e; rt = t^{-1}r; vt = t^{-1}v; rv = vr\right\rangle$
Group 7 - Has t,h and v as generators: $\left\langle t, h, r \mid h^2 = e; r^2 = e; rt = t^{-1}r; ht = th; hr = rh\right\rangle$

Note: This groups do not follow a specific order.

In figure (3) we can see better how each Frieze group works using the example of a footprint as the Motif.

References

Glassner, Andrew (1996). "Frieze groups". In: *IEEE Computer Graphics and Applications* 16.3, pp. 78–83.

Landau, Tyler (2019). "Classifications of Frieze Groups and an Introduction to Crystallographic Groups". In.

publisher, Margaret McGuireFollow this (Dec. 2017). *Frieze patterns: Approaches with group theory and Combinatorics.* URL: `https://issuu.com/idancarre/docs/mcguire_freizepatternresearchpaper`.

SHUBNIKOV, AV and VA KOPTSIK (2009). "Frieze Groups and Crystallographic Groups". In: *Contemporary Abstract Algebra*, p. 461.

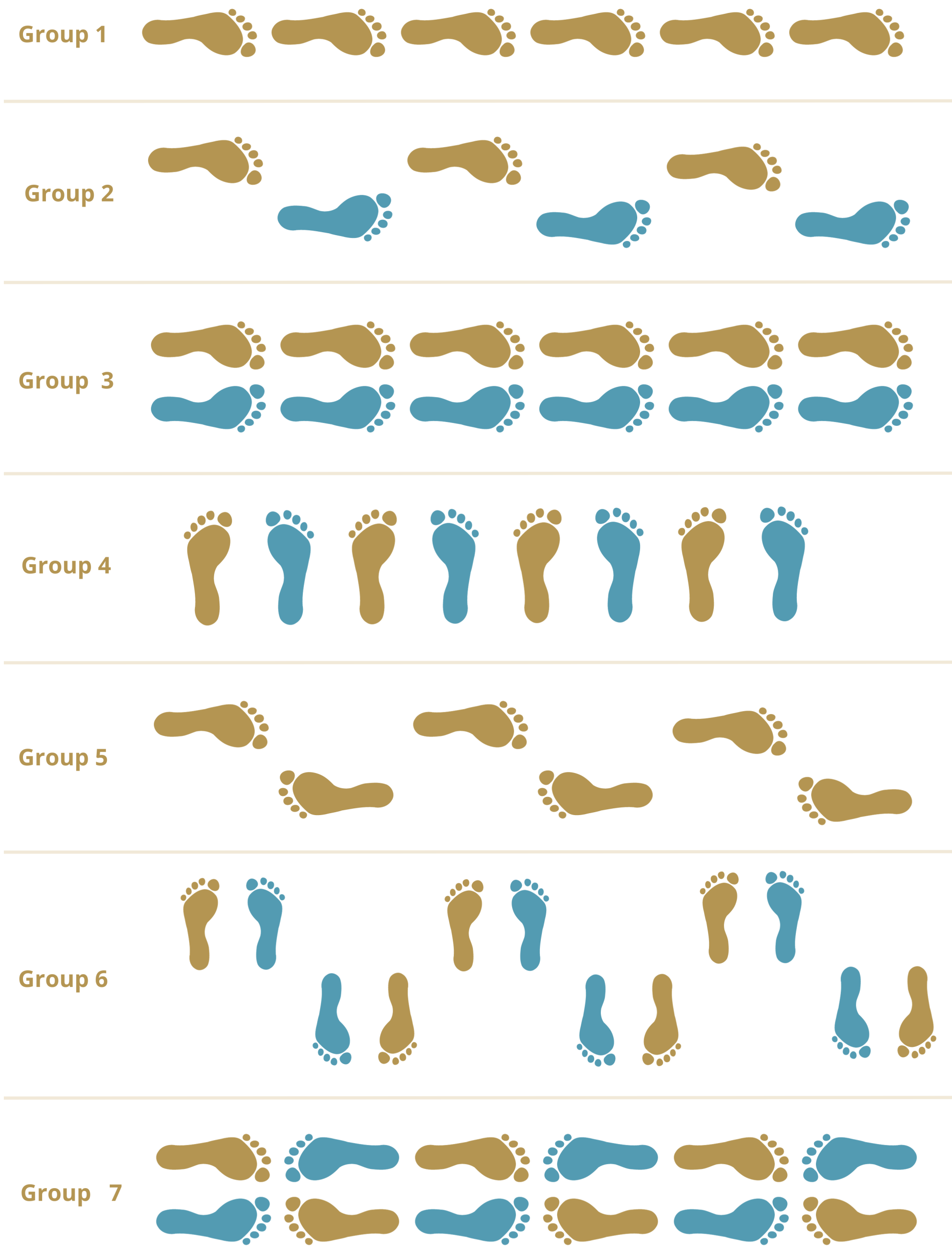

Figure 3: Example of the seven Frieze Groups using a footprint