Description of Electrodynamics with differential geometry

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1 Introduction

Let $(\mathcal{M}, g)$ be a Lorentzian manifold, where $\mathcal{M}$ is four dimensional (homeomorphic to $\mathbb{R}^4$) and $g : T_qM \times T_qM \to \mathbb{R}$ is a Lorentzian metric with the signature $(-,+,+,+)$, defined as

$$g(v,w) = -v^0w^0 + v^1w^1 + v^2w^2 + v^3w^3$$

where $v = v^0\partial_{\mathbf{0}} + v^1\partial_{\mathbf{1}} + v^2\partial_{\mathbf{2}} + v^3\partial_{\mathbf{3}}$ and $w = w^0\partial_{\mathbf{0}} + w^1\partial_{\mathbf{1}} + w^2\partial_{\mathbf{2}} + w^3\partial_{\mathbf{3}}$, with $\partial_{\mathbf{i}} = \frac{\partial}{\partial x^i}$. Notice that coordinates in $\mathcal{M}$ are denoted by (x^0, x^1, x^2, x^3), where we want to identify x^0 with "time" and the other coordinates with "space".

With the structure of $\mathcal{M}$ defined, we now want to define the electromagnetic field associated with the source 1-*form* $\boldsymbol{\mathcal{J}}$ on $\mathcal{M}$, as the 2-*form* $\boldsymbol{\mathcal{F}}$ on $\mathcal{M}$ which satisfies the *homogeneous Maxwell's equation*,

$$d\boldsymbol{\mathcal{F}} = \mathbf{0}, \tag{1}$$

and the *inhomogeneous Maxwell's equation*,

$$\delta\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{J}}. \tag{2}$$

where the operator $\delta : \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ is called the codifferential and is given by,

$$\delta = (-1)^k \star^{-1} d\star = (-1)^{n(k-1)+1} s \star d \star. \tag{3}$$

The operator $\star$ is the Hodge star operator which in $\mathcal{M}$ and in component notation is given by,

$$\star(dx^\mu) = g^{\mu\lambda}\varepsilon_{\lambda\nu\rho\sigma}\frac{1}{3!}dx^\nu \wedge dx^\rho \wedge dx^\sigma, \tag{4}$$

$$\star(dx^\mu \wedge dx^\nu) = g^{\mu\kappa}g^{\nu\lambda}\varepsilon_{\kappa\lambda\rho\sigma}\frac{1}{2!}dx^\rho \wedge dx^\sigma, \tag{5}$$

where $\varepsilon_{\lambda\nu\rho\sigma}$ is the Levi-Civita symbol with $\varepsilon_{0123} = 1$. The Hodge star of 3 and 4-*forms* can be easily deduced from the fact that, in the Lorentzian signature, $(\star)^2 = 1$ for odd-rank forms and $(\star)^2 = -1$ for even-rank forms.

2 Connection with the traditional formalism

Let's start with equation 1. We can write $\boldsymbol{\mathcal{F}}$ in components as,

$$\begin{aligned}\boldsymbol{\mathcal{F}} =& F_1 dx^0 \wedge dx^1 + F_2 dx^0 \wedge dx^2 + F_3 dx^0 \wedge dx^3 \\ &+ F_4 dx^1 \wedge dx^2 + F_5 dx^1 \wedge dx^3 + F_6 dx^2 \wedge dx^3\end{aligned}$$

and $\boldsymbol{\mathcal{J}}$ as,

$$\boldsymbol{\mathcal{J}} = J_0 dx^0 + J_1 dx^1 + J_2 dx^2 + J_3 dx^3.$$

Using the fact that $d(F_i dx^\mu \wedge dx^\nu) = dF_i \wedge dx^\mu \wedge dx^\nu$, $d\boldsymbol{\mathcal{F}}$ becomes,

$$\begin{aligned}d\boldsymbol{\mathcal{F}} =& dF_1 \wedge dx^0 \wedge dx^1 + dF_2 \wedge dx^0 \wedge dx^2 + dF_3 \wedge dx^0 \wedge dx^3 \\ &+ dF_4 \wedge dx^1 \wedge dx^2 + dF_5 \wedge dx^1 \wedge dx^3 + dF_6 \wedge dx^2 \wedge dx^3.\end{aligned} \tag{6}$$

Because $F_i = F_i(x^0, x^1, x^2, x^3)$ we have that $dF_i = \frac{\partial F_i}{\partial x^0} dx^0 + \frac{\partial F_i}{\partial x^1} dx^1 + \frac{\partial F_i}{\partial x^2} dx^2 + \frac{\partial F_i}{\partial x^3} dx^3$. Plugging this into equation 6 and rearranging the terms we get,

$$\begin{aligned} d\boldsymbol{\mathcal{F}} = &\left(\frac{\partial F_1}{\partial x^2} - \frac{\partial F_2}{\partial x^1} + \frac{\partial F_4}{\partial x^0}\right) dx^2 \wedge dx^0 \wedge dx^1 + \left(\frac{\partial F_1}{\partial x^3} - \frac{\partial F_3}{\partial x^1} + \frac{\partial F_5}{\partial x^0}\right) dx^3 \wedge dx^0 \wedge dx^1 \\ &+ \left(\frac{\partial F_2}{\partial x^3} - \frac{\partial F_3}{\partial x^2} + \frac{\partial F_6}{\partial x^0}\right) dx^3 \wedge dx^0 \wedge dx^2 + \left(\frac{\partial F_6}{\partial x^1} - \frac{\partial F_5}{\partial x^2} + \frac{\partial F_4}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{7}$$

Setting all components to zero so that equation 1 is satisfied, we have,

$$\begin{cases} \dfrac{\partial F_1}{\partial x^2} - \dfrac{\partial F_2}{\partial x^1} = -\dfrac{\partial F_4}{\partial x^0} \\ \dfrac{\partial F_1}{\partial x^3} - \dfrac{\partial F_3}{\partial x^1} = -\dfrac{\partial F_5}{\partial x^0} \\ \dfrac{\partial F_2}{\partial x^3} - \dfrac{\partial F_3}{\partial x^2} = -\dfrac{\partial F_6}{\partial x^0} \\ \dfrac{\partial F_6}{\partial x^1} - \dfrac{\partial F_5}{\partial x^2} + \dfrac{\partial F_4}{\partial x^3} = 0 \end{cases}. \tag{8}$$

We now turn to equation 2 in which we have to calculate, $\delta\boldsymbol{\mathcal{F}} = \star d \star \boldsymbol{\mathcal{F}}$. Using equations 4 and 5 we get,

$$\begin{aligned} \delta\boldsymbol{\mathcal{F}} = &-\left(\frac{\partial F_1}{\partial x^1} + \frac{\partial F_2}{\partial x^2} + \frac{\partial F_3}{\partial x^3}\right) dx^0 - \left(\frac{\partial F_5}{\partial x^3} + \frac{\partial F_4}{\partial x^2} + \frac{\partial F_1}{\partial x^0}\right) dx^1 \\ &+ \left(-\frac{\partial F_6}{\partial x^3} + \frac{\partial F_4}{\partial x^1} - \frac{\partial F_2}{\partial x^0}\right) dx^2 + \left(\frac{\partial F_6}{\partial x^2} + \frac{\partial F_5}{\partial x^1} - \frac{\partial F_3}{\partial x^0}\right) dx^3. \end{aligned}$$

Equalizing to $\boldsymbol{\mathcal{J}}$ component wise, we get the system of equations,

$$\begin{cases} \dfrac{\partial F_5}{\partial x^3} + \dfrac{\partial F_4}{\partial x^2} + \dfrac{\partial F_1}{\partial x^0} = J_1 \\ \dfrac{\partial F_6}{\partial x^3} - \dfrac{\partial F_4}{\partial x^1} + \dfrac{\partial F_2}{\partial x^0} = J_2 \\ -\dfrac{\partial F_6}{\partial x^2} - \dfrac{\partial F_5}{\partial x^1} + \dfrac{\partial F_3}{\partial x^0} = J_3 \\ \dfrac{\partial F_1}{\partial x^1} + \dfrac{\partial F_2}{\partial x^2} + \dfrac{\partial F_3}{\partial x^3} = J_0 \end{cases}. \tag{9}$$

By renaming $J_0 = -\rho$ and the components of $\boldsymbol{\mathcal{F}}$ to be,

$$\begin{cases} F_1 = -E_1 \\ F_2 = -E_2 \\ F_3 = -E_3 \\ F_4 = B_1 \\ F_5 = -B_2 \\ F_6 = B_3 \end{cases},$$

and defining the vector fields on the submanifold $\mathbf{E^3}$ (three dimensional euclidean space),

$$\begin{aligned} \mathbf{E} &= E_1\partial_1 + E_2\partial_2 + E_3\partial_3 \\ \mathbf{B} &= B_1\partial_1 + B_2\partial_2 + B_3\partial_3 \\ \mathbf{J} &= J_1\partial_1 + J_2\partial_2 + J_3\partial_3 \end{aligned}$$

equations 8 and 9 become,

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial x^0} \\ \nabla \cdot \mathbf{E} = \rho \\ \nabla \times \mathbf{B} = \mathbf{J} + \dfrac{\partial \mathbf{E}}{\partial x^0} \end{cases},$$

which are the familiar Maxwell's equations in vector calculus notation.