The Rubik's Cube Group

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Abstract

The purpose of this paper is to study and analyse the Rubik's Cube Group. We'll begin by defining the notation used along the essay, then we'll briefly define the Rubik's Cube Group, prove that it's indeed a group and study some of its properties. Finally, we'll compute the order of the group.

I. Introduction

A Rubik's Cube is a worldwide popular three-dimensional combination puzzle that was invented in 1974 by Hungarian sculptor and professor of architecture Ernő Rubik.

The puzzle consists of a cube with six faces, each made up of nine smaller squares of a single colour called cubies. The faces can be twisted and turned in any direction, and the end goal of the puzzle is to rearrange the smaller squares so that each face of the cube is a single colour, meaning all cubies on each face have the same colour.

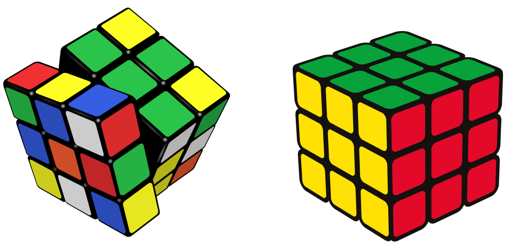

Figure 1: Unsolved and Solved Rubik's Cubes

II. Notation

First, let's describe the cube's moves with the *Singmaster* notation. That is, say that our cube is sitting on a flat surface and each clockwise turn of the face is a rotation by 90 degrees clockwise. Then, the cube's basic moves can be denoted by the elements F, B, U, D, L and R as follows:

$$\begin{cases} F: & \text{turns the front face clockwise} \\ B: & \text{turns the back face clockwise} \\ U: & \text{turns the top/upward face clockwise} \\ D: & \text{turns the bottom/downward face clockwise} \\ L: & \text{turns the left face clockwise} \\ R: & \text{turns the right face clockwise} \end{cases} \tag{1}$$

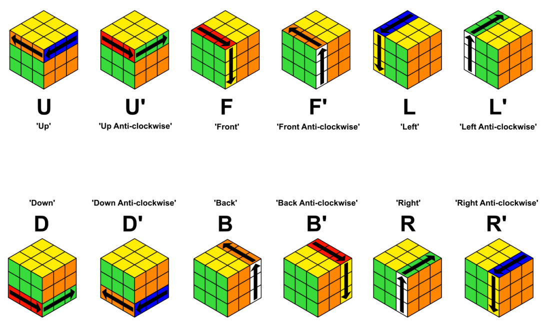

Figure 2: Rubik's cube basic moves

Remark: The move E can be thought of as the identity element, meaning the cube remains in the same position. Then, $LLLL = L^4 = E$ and $RRR = R'$, where R', the inverse of R, turns the right face counterclockwise.

III. The Rubik's Cube Group

I Construction of the group

Note that there are 54 cubies on the Rubik's Cube that can be arranged and rearranged through twisting and turning the faces. Any position of the cube can be described as a permutation. Therefore, the Rubik's Cube group $\mathbb{G}$ is a subgroup of the permutation group of 54 elements S_{54}. However, we can easily see that no matter the permutation, the center cubies remain in the same position. Thus, $\mathbb{G}$ is a subgroup of S_{48}.

Let $(\mathbb{G}, \cdot)$ be the Rubik's Cube Group. Then :

1. $\mathbb{G}$ is the set of all possible moves.

2. $M_1 \cdot M_2$ is the move where you first do M_1 and then M_2.

3. Two moves are said to be the same if they result in the same configuration.

4. We can represent any cube configuration by detailing the sequence of moves from the start position to the that position, so all moves are sequences of one or more basic moves. Therefore, $\mathbb{G}$ can be generated by all basic moves, meaning $\mathbb{G} =< F, B, U, D, L, R >$.

Proof that it is a group:

1. It's <u>associative</u> : $M_1 \cdot (M_2 \cdot M_3) = (M_1 \cdot M_2) \cdot M_3$, $\forall M_1, M_2, M_3 \in \mathbb{G}$. It suffices to look at the fact that $M_1 \cdot (M_2 \cdot M_3) = M_1(M_2(M_3(Cubie)) = (M_1 \cdot M_2) \cdot M_3$.

2. There is an <u>identity element</u> $E \in \mathbb{G}$ such that $M \cdot E = E \cdot M = E$, $\forall M \in \mathbb{G}$. As mentioned previously, $MMMM = M^4 = E$.

3. $\forall M \in \mathbb{G}$ there exists $M' \in \mathbb{G}$ called the <u>inverse</u> of M such that $M \cdot M' = M' \cdot M = E$.

   $(M_1 \cdot M_2)' = (M_2)' \cdot (M_1)'$, $\forall M_1, M_2 \in \mathbb{G}$ and as we saw in the previous section, $M' = MMM = M^3$, $\forall M \in \mathbb{G}$.

Remark: Doing two sequences of cube moves in a different order can result in a different configuration, meaning a composition of cube moves is not commutative. Thus, $\mathbb{G}$ is **non-abelian**.

II Valid configurations

Any configuration of the Rubik's Cube can be described by an ordered 4-tuple (v, r, w, s) such that :

$$v \in (\mathbb{Z}/3\mathbb{Z})^8, r \in S_8, w \in (\mathbb{Z}/2\mathbb{Z})^{12}, \text{and} s \in S_{12} \tag{2}$$

However, not all of these configurations are valid. In this sense, we have the First Fundamental Theorems of Cube Theory :

I Fundamental Theorem of Cube Theory:

Let $v \in (\mathbb{Z}/3\mathbb{Z})^8, r \in S_8, w \in (\mathbb{Z}/2\mathbb{Z})^{12}$, and $s \in S_{12}$. The 4-tuple (v, r, w, s) corresponds to a possible configuration of the cube if and only if:

1. Equal parity of permutations : $sgn(r) = sgn(s)$.

2. Conservation of the total number of twists : $v_1 + v_2 + v_3 + ... + v_8 \equiv 0 \mod 3$.

3. Conservation of the total number of flips : $w_1 + w_2 + w_3 + ... + w_{12} \equiv 0 \mod 2$.

III Order of the group

After learning that not all of the configurations are valid, using the IFirst Fundamental Theorem of Cube Theory, we can compute the order of the group :

$$|\mathbb{G}| = \frac{12! \times 8!}{2} \times \frac{3^8}{3} \times \frac{2^{12}}{2} \tag{3}$$

Where $|S_{12}| = 12!$, $|S_8| = 8!$, $|(\mathbb{Z}/3\mathbb{Z})^8| = 3^8$ and $|(\mathbb{Z}/2\mathbb{Z})^{12}| = 2^{12}$.

IV. Conclusion

In short, Group Theory allows us to simplify this puzzle and study many of its properties. Using Group Theory, it's possible to construct an algorithm to solve any Rubik's cube. It's also possible to compute the minimum number of moves required to solve any configuration of the puzzle, regardless of how scrambled it is. This number is known as **God's number for the Rubik's Cube** and, in July 2010, it was proven that God's number for the standard 3x3 cube is **20 in the half-turn metric** (where a half-twist is counted as a single move) or **26 in the quarter-turn metric** (where a half-twist is counted as two quarter-twists). This shows how useful Group Theory can be when it comes to solving what would appear to be a difficult puzzle otherwise.

References

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