Representations of SU_3 in Physics

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Abstract

A summary of the representation theory of SU_n is given, with special focus on SU_3 and its Lie algebra representations. Essential applications of this theory to particle physics are detailed, accompanied by a historical contextualization.

1 Representations of SU_3

1.1 Representations of Lie Groups

Let us begin by discussing the algebraic and topological structure of $SU_n,\ n \geq 2$. As a set, it consists of all $n \times n$ unitary matrices with determinant 1. It is a subgroup of $GL_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$ which is closed and bounded, thus compact. From Cartan's closed subgroup theorem, it is also a submanifold of $GL_n(\mathbb{C})$ and a (matrix) Lie group, of dimension $n^2 - 1$ (which we'll prove later). Furthermore, it is simply connected. This can be proved by considering the action $SU_n \overset{\phi}{\curvearrowright} S^{2n-1}$ (Prop. 13.11, in [1]). The action is transitive, and the stabilizer of each point $p \in S^{2n-1}$ is $G_p = SU_{n-1}$ and so SU_n is a fiber bundle with base S^{2n-1} and fiber $SUn - 1$. By a standard topological argument, we can then deduce that $\pi_1(SU_n) \cong \pi_1(SU_{n-1})$, and, since $\pi_1(SU_2) \cong \pi_1(S^3)$ is trivial, by induction, we get the result.

The Lie algebra of SU_n, denoted by $\mathfrak{su}_n \cong T_{I_n}SU_n$, can be identified as the algebra of the $n \times n$ antihermitian, traceless matrices, by differentiating the exponential map. Like for any matrix Lie algebra, the left-invariant vector field commutator bracket in $\mathfrak{X}_L(G)$ descends to the matrix commutator bracket in $\mathfrak{g}$ to endow it with a Lie algebra structure. It is straightforward then to see that $\dim_{\mathbb{R}} \mathfrak{su}_n = n^2 - 1$, which is thus the same real dimension as the Lie group SU_n.

Now, the power of Lie theory comes from the fact that we can study the Lie group and its representations through their linearization on the Lie algebra. This is evidenced through a series of results collectively called the *Lie group-Lie algebra correspondence*. Firstly, we should note any finite-dimensional representation of the Lie group $\phi : G \to GL_n(\mathbb{C})$ induces a Lie algebra representation through its pushforward $\phi_* : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$. The subtle point of this correspondence is thus the converse of the previous assertion. The two most important theorems are:

Theorem 1.1 (Homomorphism theorem, 5.6 [1])**.** *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ resp., and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\Phi(e^X) = e^{\phi(X)},\ \forall X \in \mathfrak{g}$.*

Theorem 1.2 (Lie's third theorem, 5.25 [1])**.** *If $\mathfrak{g}$ is a finite-dimensional real Lie algebra, there exists a connected Lie subgroup $G \subset GL_n(\mathbb{C})$ whose Lie algebra is isomorphic to $\mathfrak{g}$.*

The first theorem is proved by inverting the exponential map (which is surjective for compact Lie groups) via the Baker-Campbell-Hausdorff formula. The second is a consequence of Ado's theorem, which asserts that every finite-dimensional real (or complex) Lie algebra is isomorphic to a matrix Lie algebra. In tandem, these two results allow us to create a bijection between the finite-dimensional representations of a matrix Lie group and the ones of its Lie algebra, and so classifying the former reduces to classifying the latter.

1.2 Representations of Lie algebras

In order to better study the representations of the real Lie algebra $\mathfrak{g} = \mathfrak{su}_3$, we will rely heavily on the following theorem:

Proposition 1.3 (4.6 [1])**.** *Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification, then every finite-dimensional complex representation ϕ of $\mathfrak{g}$ has a unique extension to a complex-linear representation of $\mathfrak{g}_{\mathbb{C}}$. Furthermore, ϕ is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$ if and only if it is irreducible as a representation of $\mathfrak{g}$.*

Moreover, using "Weyl's unitary trick", we can prove that every finite-dimensional representation of a compact Lie group, and so its Lie algebra, is completely reducible. We can thus restrict our attention to the irreducible representations of the complexification $\mathfrak{g}_{\mathbb{C}}$, since, by restriction, they induce finite-dimensional representations on the real form $\mathfrak{g}$, and thus on the Lie group G. For the case of $\mathfrak{su}_n$, the complexification can be readily seen to be the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of $n \times n$ traceless matrices, which is simple (ie. has no non-trivial ideals), but in general, it might only be semi-simple (ie. a direct sum of simple Lie algebras). The following discussion is valid also for semi-simple Lie algebras, so we'll stay in that context.

A complex semi-simple Lie algebra $\mathfrak{g}$ contains a unique maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that ad_H are simultaneously diagonalizable $\forall H \in \mathfrak{h}$, called a *Cartan subalgebra*, whose (complex) dimension is the *rank* r of $\mathfrak{g}$. Given a representation (ϕ, V) of $\mathfrak{g}$, a *weight* λ for that representation is a linear functional $\lambda : \mathfrak{h} \to \mathbb{C}$ such that $\phi(H)v = \lambda(H)v,\ \forall H \in \mathfrak{h}, \forall v \neq 0 \in V$. For the adjoint representation $\text{ad} : g \mapsto \text{ad}_g$, the weights are called *roots*. The Cartan subalgebra has a real subspace $\mathfrak{h}_{\mathbb{R}}$ of (real) dimension equal to the rank, which inherits an inner product structure from the *Killing form* of $\mathfrak{g}$. The dual of this space is spanned by the r roots, and we can represent the roots and weights in a lattice $\Lambda \subset \mathbb{R}^r$ (see Fig. 1). We can define a partial order on the weights, which allows us to state the central theorem of this section:

Theorem 1.4 (Theorem of the highest weight, 9.4 and 9.5 [1])**.** *Every irreducible (finite-dimensional) representation of a semi-simple complex Lie algebra $\mathfrak{g}$ has a highest weight λ_{max} (which is always dominant and algebraically integral), and conversely, every dominant, algebraically integral element is the highest weight of an irreducible representation. Furthermore, two irreducible representations with the same highest weight are isomorphic.*

This exceptional theorem fully classifies all the finite-dimensional representations via their highest weight. The hardest part of the proof is indeed in the converse, since it is non-elementary to construct the representations in question. In the case of $\mathfrak{sl}_n(\mathbb{C})$, there is a simple construction we will highlight in the next section, using the tensor product and the Clebsch-Gordan decomposition.

For the 8-dimensional Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ (of rank 2), we can work with the basis of the Cartan algebra:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1}$$

An irreducible representation ϕ is then classified by the largest eigenvalues $m_1,\ m_2 \in \mathbb{N}$ of $\phi(H_1),\ \phi(H_2)$, resp., and denoted by $D(m_1, m_2)$ (the respective roots are α and β in Fig. 1). All the weights of the representation can be reached from the highest weight by subtracting multiples of the roots, in sequences called α-strings (important examples can be seen in Fig. 2 and 3). These representations have dimensions given by:

$$d(m_1, m_2) = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2) \tag{2}$$

In the next section, we will discover how these results are take central stage in particle physics.

2 Application in Physics

2.1 Symmetries and Isospin

In quantum mechanics, two states of a Hilbert space H that are multiples of each other by a complex number are identified, thus creating a projective Hilbert space $\mathbf{P}(H)$ called the *ray space*. In 1931, at the onset of the mathematical formulation of quantum mechanics, Wigner proved a fundamental theorem of the mathematical formulation of quantum mechanics [2]:

Theorem 2.1 (Wigner, 1931 [3])**.** *If $T : \mathbf{P}(H) \to \mathbf{P}(H')$ is an isometry, then there exists either a linear and unitary or antilinear and antiunitary isometry $U : H \to H'$ which represents T, up to a phase factor.*

If the symmetry group $G \ni T$ is (semi)simple and simply connected, then the phase factor can be taken to be always one. This suggests that the (Lie) groups U_n have a special place in quantum mechanics as the allowed symmetries of the theory.

One year later, Heisenberg introduced the concept of *isospin* to explain why the proton and neutron were affected equally by the strong force despite having different charges. He proposed the strong interaction had a (approximate) flavor symmetry under the action of the Lie group SU_2 (you might ask, why not U_2? see footnote[1]). This was corroborated by the fact that the mesons appeared in triplet $\mathbf{3}$ and singlet $\mathbf{1}$ representations, like for the SU_2 angular momentum (described by $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$ for the mesons, $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}$ for the baryons).

With all we have said here, one could explicitly describe the up and down isospin components and find their charges (the up and down quarks!), if one had the insight, but already in the 50s, the isospin model was insufficient to fully characterize the "particle zoo", since it failed to account for the property of *strangeness* (strange quark content). This prompted Gell-Mann and Ne'eman to independently suggest in 1961 to extend the flavor symmetry group to SU_3 [4]. The mesons were organized into octets and singlets, and the baryons into octets and decuplets. Only afterwards, in 1964, did Gell-Mann and Zweig independently realize these groupings of particles were a consequence of combining the quarks in SU_3 flavor symmetric representations. In the quark model, the mesons could be better interpreted as a nonet: $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ (see Fig. 4). For baryons, we get $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ (see Fig. 5). An important remark is that these flavor symmetries are not exact since the mass of the up, down, and strange quarks are slightly different (the strange quark is quite a bit more massive), and there are other symmetry breaking subtleties we are not addressing here.

Lastly, the study of the Δ^{++} led to the discovery of another SU_3 symmetry, this one exact: the *color charge*. The spin $\frac{3}{2}$ Δ^{++} baryon required three up quarks with parallel spins and vanishing orbital angular momentum, which would violate the Pauli exclusion principle. Han and Nambu suggested the existence of this hidden degree of freedom which is intertwined with the flavor symmetry. Since color cannot be observed, the only physical representations are the ones which contain a singlet (or trivial) representation, ie. $\mathbf{3} \otimes \bar{\mathbf{3}}$, $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$, $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \bar{\mathbf{3}}$, etc. For this symmetry, the quarks transform in the fundamental representation, the antiquarks in the antifundamental, and the 8 gluons in the adjoint.

It is noteworthy to elaborate on the construction alluded to above. Let's take the example of $\mathfrak{sl}_3(\mathbb{C})$ (works similarly for any $\mathfrak{sl}_n(\mathbb{C})$). First, we construct the fundamental representations, consisting of the standard representation $\phi = Id$ (fundamental $\mathbf{3}$, $\lambda_{max} = (1, 0)$) and the *dual* $\phi(X) = -X^t$ (antifundamental $\bar{\mathbf{3}}$, $\lambda_{max} = (0, 1)$). We can then tensor product m_1 times $\mathbf{3}$ and m_2 times $\bar{\mathbf{3}}$ and then extract the invariant subspaces (Prop. 6.17 [1]): this process creates all irreducible representations since it creates all highest weights λ_{max}.

We conclude by mentioning a powerful method for decomposing the above tensor products of SU_n group representations into a sum of irreducible representations: using Young tableaux, a useful combinatorics tool. An example can be found in Fig. 6.

References

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[4] Pierre Ramond. *Group Theory: A Physicist's Survey*. Cambridge University Press, 2010.

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[1] The symmetry groups are almost always chosen to be SU_n instead of U_n for the simple reason that $U_n \supset SU_n \times U_1$ and so the theory would have an extra particle that would transform with the trivial representation $\mathbf{1}$, ie. not transform. This is contradicted by experiment in the Standard Model (such as there being no 9th, color-neutral, gluon).

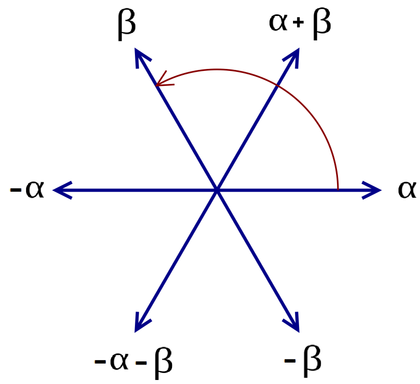

Figure 1: Root system of $\mathfrak{sl}_3(\mathbb{C})$ (A_2).

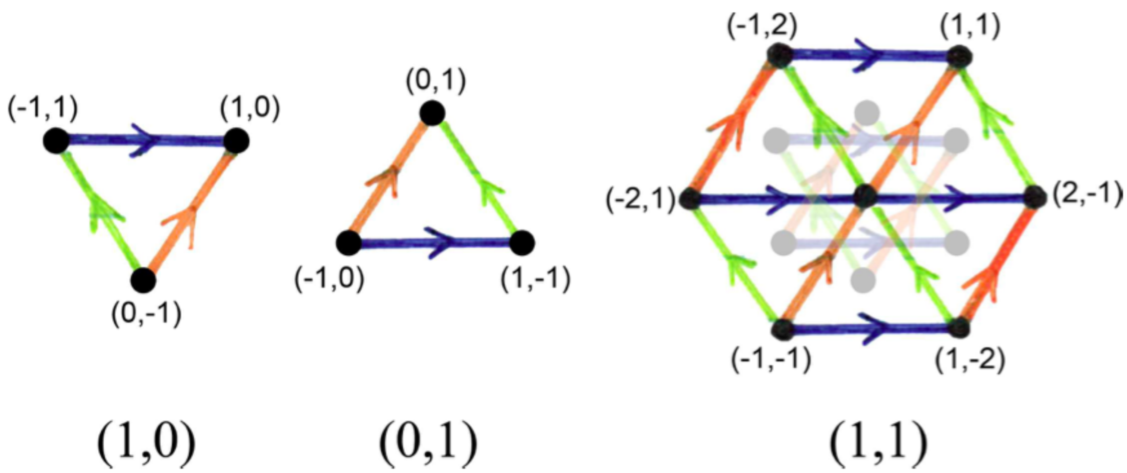

Figure 2: Weight lattices of the fundamental ($\mathbf{3}$, $(1,0)$), antifundamental ($\bar{\mathbf{3}}$, $(0,1)$) and adjoint ($\mathbf{8}$, $(1,1)$) representations of $\mathfrak{sl}_3(\mathbb{C})$. The arrows correspond to the roots: blue for α, green for β, orange for $\alpha+\beta$.[7]

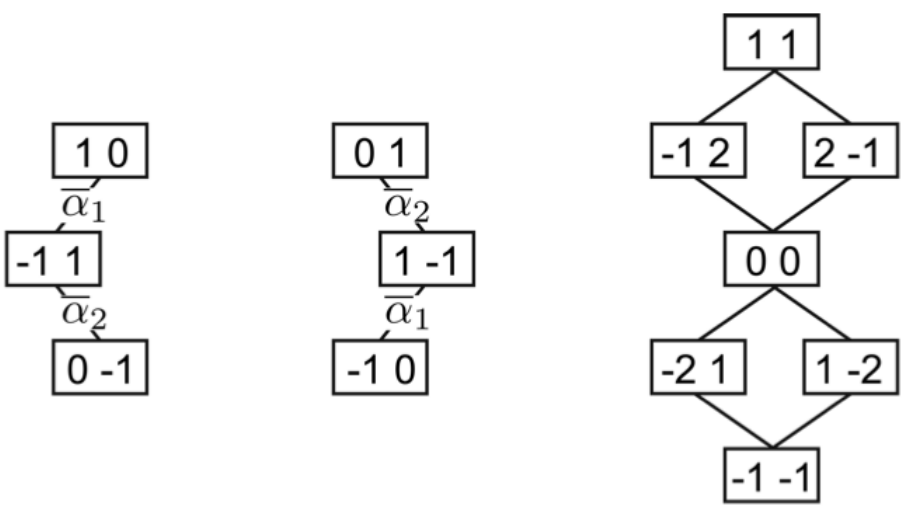

Figure 3: Weight graphs of the representations in Fig. 2.[7]

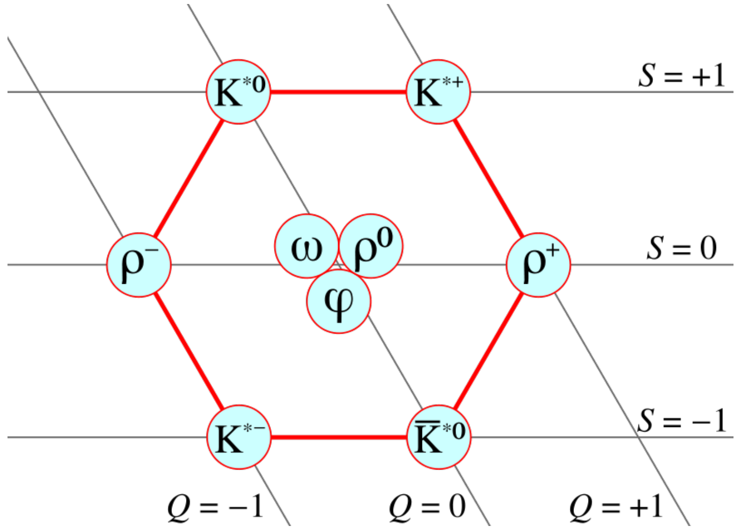

Figure 4: Nonet of spin 1 mesons.

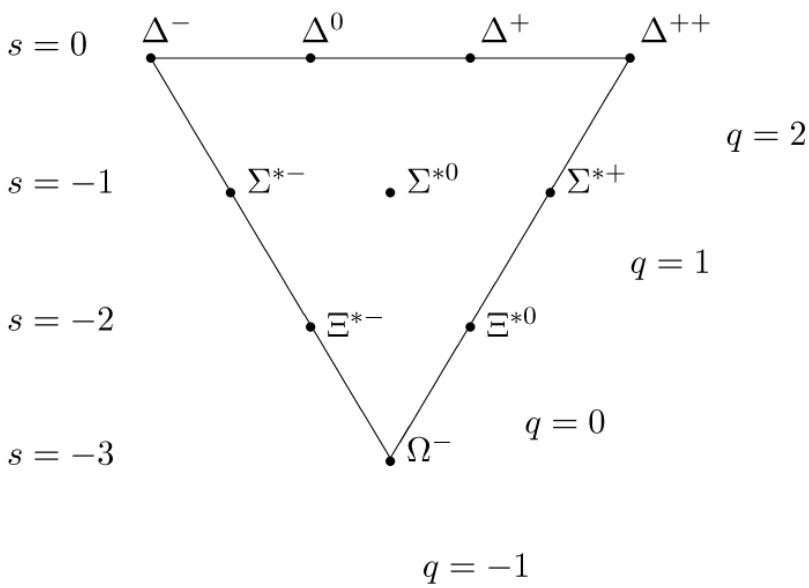

Figure 5: Decuplet of spin $\frac{3}{2}$ baryons.

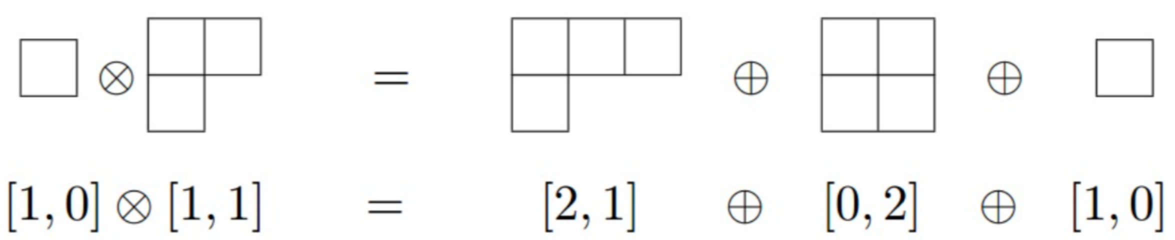

Figure 6: Young diagrams of the decomposition $D(1,0) \otimes D(1,1) = D(2,1) \oplus D(0,2) \oplus D(1,0)$.