Algebraic and Geometric Methods in Engineering and Physics

Homework 11

Due on December 13

1. Consider the subspace topology on the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

the open sets

$$U_N = S^2 \setminus \{(0,0,1)\} \qquad \text{and} \qquad U_S = S^2 \setminus \{(0,0,-1)\},$$

and the homeomorphisms $\varphi_N : U_N \to \mathbb{R}^2$ and $\varphi_S : U_S \to \mathbb{R}^2$ given by

$$\varphi_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right) \qquad \text{and} \qquad \varphi_S(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right).$$

Show that $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is an atlas for S^2 (which is therefore a 2-dimensional differentiable manifold).

2. Compute all possible Lie brackets of the vector fields $X, Y, Z \in \mathfrak{X}(\mathbb{R}^3)$ given by

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\,, \qquad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\,, \qquad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\,.$$