Algebraic and Geometric Methods in Engineering and Physics

Homework 10

Due on December 6

1. Consider the family of subsets of $\mathbb{R}$ given by

$$\mathcal{T}^{\mathrm{semi}} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}.$$

   Show that:

   (a) $\mathcal{T}^{\mathrm{semi}}$ is a topology on $\mathbb{R}$.

   (b) $K \neq \varnothing$ is compact for this topology if and only if it has maximum.

   (c) $f : M \to \mathbb{R}$ is a continuous map between the topological spaces $(M, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}^{\mathrm{semi}})$ if and only if for every $x \in M$ and every $\varepsilon > 0$ there exists $U \in \mathcal{T}$ such that $x \in U$ and $f(y) < f(x) + \varepsilon$ for every $y \in U$.

   (**Remark:** These functions are sometimes called **upper semicontinuous**. Therefore, upper semicontinuous functions always have maxima on compact sets).