Algebraic and Geometric Methods in Engineering and Physics 2023/2024

2^{nd} Exam - 1 February 2024 - 10:30
Duration: 2 hours

(6/20) **1.** The **commutator subgroup** of a group G is the smallest subgroup $H \subset G$ which contains all elements of the form $ghg^{-1}h^{-1}$ for $g, h \in G$ (that is, it is the intersection of all subgroups with this property). Show that:

(a) H is a normal subgroup.

(b) G/H is abelian.

(c) Each irreducible representation of G/H determines a 1-dimensional representation of G.

(d) Every 1-dimensional representation of G can be obtained in this way.

(3/20) **2.** The commutator subgroup of the alternating group A_4 (that is, the group of even permutations of $\{1, 2, 3, 4\}$) is the Klein group $K = \{e, (12)(34), (13)(24), (14)(23)\}$. How many irreducible representations does A_4 have? What are their dimensions?

(4/20) **3.** Solve the equation $x^{2022} = 5$ in $\mathbb{Z}_{11}$.

(3/20) **4.** Let (M, d) be a metric space and $K \subset M$ a compact set.

(a) Show that K is necessarily closed and bounded (that is, contained in some open ball).

(b) Show that in the metric space $(\mathbb{R} \setminus \{0\}, d)$, where $d(x, y) = |x - y|$ is the usual distance function, there are closed bounded sets which are not compact.

(4/20) **5.** The **Heisenberg algebra** is the Lie algebra $\mathfrak{h}$ generated by the matrices

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the **Heisenberg group** is

$$H = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : x, y, z \in \mathbb{R} \right\}.$$

Show that:

(a) The Lie brackets of X, Y and Z are all zero except for $[X, Y] = Z$. Is the Heisenberg algebra simple?

(b) H is a Lie group.

(c) $\mathfrak{h}$ is the Lie algebra of H.