Algebraic and Geometric Methods in Engineering and Physics

2023/2024

1^{st} Exam - 18 January 2024 - 10:30
Duration: 2 hours

(9/20) **1.** Consider the finite group $G = \mathbb{Z}_2 \times S_3$.

(a) Show that G is a nonabelian group of order 12.

(b) Is this group isomorphic to the rotation group of the tetrahedron? Why or why not?

(c) Prove that $H = \mathbb{Z}_2 \times \{e\}$ is a normal subgroup, and identify the quotient group G/H.

(d) Show that each irreducible representation of either $\mathbb{Z}_2$ or S_3 determines an irreducible representation of G. How many irreducible representations of G can one obtain in this way?

(e) List the conjugation classes of G.

(f) How many irreducible representations does G have? What are their dimensions?

(4/20) **2.** Let $p > 2$ be a prime number. If we use n colors, how many colorings are there of the vertices of a polygon with p sides, up to rotations of the polygon? Use your answer to prove that $n^p - n$ is a multiple of p (**Fermat's Little Theorem**).

(3/20) **3.** Consider the **cofinite topology** in $\mathbb{R}$, that is, the topology whose closed sets are the finite sets (and $\mathbb{R}$).

(a) Show that any injective function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

(b) Give an example of a disconnected set.

(4/20) **4.** Consider the Lie group

$$SU(2) = \{S \in \mathcal{M}_{2\times 2}(\mathbb{C}) : S^*S = I \text{ and } \det(S) = 1\},$$

whose Lie algebra is

$$\begin{aligned} \mathfrak{su}(2) &= \{A \in \mathcal{M}_{2\times 2}(\mathbb{C}) : A^* = -A \text{ and } \operatorname{tr}(A) = 0\} \\ &= \left\{ \begin{bmatrix} iz & x+iy \\ -x+iy & -iz \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C}) : x, y, z \in \mathbb{R} \right\}. \end{aligned}$$

Show that:

(a) $\mathfrak{su}(2)$ is simple.

(b) $\varphi_S(A) = SAS^{-1}$ defines an action of $SU(2)$ on $\mathfrak{su}(2)$.

(c) If we identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ by using the coordinates (x, y, z) then this action is a homomorphism $\varphi : SU(2) \to SO(3)$ with kernel $\{I, -I\}$.