Geometria Riemanniana

Ficha 6

A entregar até à aula de Terça-feira dia 31 de Outubro

1. Seja V um espaço vectorial de dimensão n. Mostre que $\theta^1, \ldots, \theta^k \in V^*$ são linearmente independentes **sse** $\theta^1 \wedge \ldots \wedge \theta^k \neq 0$.

2. Considere as seguintes formas diferenciais:

$$\begin{aligned}
&\alpha = xdx + ydy \in \Omega^1(\mathbb{R}^2);\\
&\beta = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(\mathbb{R}^2 \setminus \{0\});\\
&\omega = e^{xz}dx + x\cos z dy + y^2 dz \in \Omega^1(\mathbb{R}^3);\\
&\eta = xdx \wedge dy - zdx \wedge dz + xyzdy \wedge dz \in \Omega^2(\mathbb{R}^3);\\
&\zeta = dx^1 \wedge dx^2 + \ldots + dx^{2n-1} \wedge dx^{2n} \in \Omega^2(\mathbb{R}^{2n}).
\end{aligned}$$

   Considere ainda as seguintes funções C^∞:

$$\begin{aligned}
&f : \mathbb{R} \to \mathbb{R}^2 \text{ definida por } f(t) = (t, t^2);\\
&g :]0, +\infty[\times]0, 2\pi[\to \mathbb{R}^2 \text{ definida por } g(r, \theta) = (r\cos\theta, r \operatorname{sen}\theta);\\
&h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ definida por } h(u, v, w) = (uv, vw, uw).
\end{aligned}$$

   Calcule:

   (a) $\alpha \wedge \beta$, $\omega \wedge \eta$, $\eta \wedge \eta$;
   (b) $\zeta \wedge \ldots \wedge \zeta$ (produto exterior com n factores);
   (c) $d\alpha, d\beta, d\omega, d\eta, d\zeta$.
   (d) $f^*\alpha, g^*\alpha, g^*\beta, h^*\eta$

   Não precisam de entregar:

3. Considere os isomorfismos vectoriais $i_1 : \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ e $i_2 : \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ dados por

$$\begin{aligned}
&i_1\left(X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} + X^3\frac{\partial}{\partial z}\right) = X^1dx + X^2dy + X^3dz;\\
&i_2\left(X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} + X^3\frac{\partial}{\partial z}\right) = X^1dy \wedge dz + X^2dz \wedge dx + X^3dx \wedge dy.
\end{aligned}$$

   Mostre que:

   (a) $df = i_1(\nabla f)$, onde $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ é o **gradiente** da função $f \in C^\infty(\mathbb{R}^3)$;

(b) $d(i_2(X)) = (\nabla \cdot X)\, dx \wedge dy \wedge dz$, onde $\nabla \cdot X = \dfrac{\partial X^1}{\partial x} + \dfrac{\partial X^2}{\partial x} + \dfrac{\partial X^3}{\partial z}$ é a **divergência** de X;

(c) $d(i_1(X)) = i_2(\nabla \times X)$, onde $\nabla \times X = \left(\dfrac{\partial X^3}{\partial y} - \dfrac{\partial X^2}{\partial z}, \dfrac{\partial X^1}{\partial z} - \dfrac{\partial X^3}{\partial x}, \dfrac{\partial X^2}{\partial x} - \dfrac{\partial X^1}{\partial y}\right)$ é o **rotacional** de X;

(d) $\nabla \times (\nabla f) = 0$;

(e) $\nabla \cdot (\nabla \times X) = 0$.