Lie Groups and Lie Algebras

2012/2013

2nd Test - 20 December 2012 - 14:30

(4/20) **1.** Let $\mathfrak{g}$ be a semisimple Lie algebra. Show that $\mathfrak{g}$ is necessarily isomorphic to a matrix Lie algebra.

Remark: This result is actually true for any Lie algebra (*Ado's Theorem*).

(4/20) **2.** Use the Weyl dimension formula for the dimension of a representation with highest weight μ,

$$\dim(\pi) = \frac{\Pi_{\alpha \in R^+} \langle \alpha, \mu + \delta \rangle}{\Pi_{\alpha \in R^+} \langle \alpha, \delta \rangle}$$

(where R^+ is the set of positive roots and δ is half the sum of the positive roots) to obtain the dimension of a representation of $\mathfrak{sl}(3, \mathbb{C})$ with highest weight $\mu = m_1\mu_1 + m_2\mu_2$ (where μ_1, μ_2 are the fundamental weights):

$$\dim(\pi) = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2).$$

3. Let $\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{6}, \mathbf{8}$ and $\mathbf{10}$ represent the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$ with highest weights $(0, 0)$ (trivial), $(1, 0)$ (fundamental), $(0, 1)$ (dual of the fundamental), $(2, 0)$, $(1, 1)$ (adjoint) and $(3, 0)$ (note that the numbers in bold give the dimensions of the corresponding representations). Show that:

(4/20) (a) $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$;

(4/20) (b) $\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$;

(4/20) (c) $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$.

Hint: Use the weight vectors of the fundamental representation $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$ to contruct highest weight vectors inside the tensor products.