Differential Geometry of Curves and Surfaces

Homework 7

Due on November 15

1. Let $\mathbf{c} : [a,b] \to \mathbb{R}^2$ be a closed simple curve enclosing a bounded open set $U \subset \mathbb{R}^2$. Use the Stokes Theorem to prove that the area of U is

$$A = \pm \int_a^b \frac{1}{2} \det(\mathbf{c}(t), \dot{\mathbf{c}}(t)) dt,$$

where the sign depends on the orientation of $\mathbf{c}$.

2. Compute the first and second fundamental forms, the mean curvature and the Gauss curvature of the following surfaces:

   (a) The cylinder $S_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = R^2\}$, using the parameterization $\mathbf{g}_1 : (0, 2\pi) \times \mathbb{R} \to S_1$ given by $\mathbf{g}_1(\varphi, z) = (R\cos\varphi, R\sin\varphi, z)$;

   (b) The cone $S_2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, z > 0\}$, using the parameterization $\mathbf{g}_2 : (0, 2\pi) \times \mathbb{R}^+ \to S_2$ given by $\mathbf{g}_2(\varphi, z) = (z\cos\varphi, z\sin\varphi, z)$;

   (c) The catenoid $S_3 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = \cosh^2 z\}$, using the parameterization $\mathbf{g}_3 : (0, 2\pi) \times \mathbb{R} \to S_3$ given by $\mathbf{g}_3(\varphi, z) = (\cosh z \cos\varphi, \cosh z \sin\varphi, z)$;

   (d) The sphere $S_4 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2\}$, using the parameterization $\mathbf{g}_4 : (0, \pi) \times (0, 2\pi) \to S_4$ given by $\mathbf{g}_4(\theta, \varphi) = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)$.