Differential Geometry of Curves and Surfaces

Homework 6

Due on October 25

1. Consider the 3-dimensional manifold with boundary

$$M = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 2 \wedge x^2 + y^2 \leq 1\}$$

and the 2-form

$$\omega = (-ydx + xdy) \wedge (-zdw + wdz) \in \Omega^2(\mathbb{R}^4).$$

(a) Check that $\mathbf{g} : (0, \frac{\pi}{4}) \times (0, 2\pi) \times (0, 2\pi) \to M$ given by

$$\mathbf{g}(\theta, \varphi, \psi) = (\sqrt{2}\sin\theta\cos\varphi, \sqrt{2}\sin\theta\sin\varphi, \sqrt{2}\cos\theta\cos\psi, \sqrt{2}\cos\theta\sin\psi)$$

is a parameterization of M.

(b) Compute

$$\int_M d\omega$$

for your choice of orientation of M by using:

(i) The definition of integral on M.

(ii) The Stokes Theorem.

2. Consider the compact 1-dimensional manifold

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

oriented in the anti-clockwise direction, and the 1-form

$$\omega = -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0, 0)\}).$$

Show that:

(a) $d\omega = 0$.

(b) $\displaystyle\int_{S^1} \omega = 2\pi$.

(c) ω is not exact.

(d) $\mathbb{R}^2 \setminus \{(0, 0)\}$ is not diffeomorphic to $\mathbb{R}^2$.