Differential Geometry of Curves and Surfaces

Homework 10

Due on December 6

1. Consider the Riemannian metric defined on $\mathbb{R}^+ \times (0, 2\pi)$ by

$$ds^2 = dr^2 + f^2(r)d\theta^2.$$

Show that:

(a) This Riemannian metric can be written as

$$ds^2 = \left(\theta^1\right)^2 + \left(\theta^2\right)^2,$$

where

$$\theta^1 = dr, \qquad \theta^2 = f(r)d\theta.$$

(b) The orthonornormal frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ dual to $\{\theta^1, \theta^2\}$ is given by

$$\mathbf{e}_1 = \frac{\partial}{\partial r}, \qquad \mathbf{e}_2 = \frac{1}{f(r)}\frac{\partial}{\partial \theta}.$$

(c) The connection form associated to $\{\theta^1, \theta^2\}$ is

$${\omega_2}^1 = -f'(r)dx.$$

(d) The Gauss curvature of this Riemannian surface is

$$K(r, \theta) = -\frac{f''(r)}{f(r)}.$$

(e) Interpret the result of the previous question in the cases $f(r) = r$ and $f(r) = \sin(r)$. What do you think is the Riemannian surface corresponding to the case $f(r) = \sinh(r)$?