Differential Geometry of Curves and Surfaces
2024/2025

2^{nd} Exam - 3 February 2025 - 10:30
Duration: 2 hours

(2/20) **1.** A regular space curve $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is called a **generalized helix** if its tangent vector makes a constant angle $\theta \in (0, \pi)$ with a fixed nonzero vector $\mathbf{v}$. Prove that the if $\mathbf{c}$ is a generalized helix then its curvature κ and its torsion τ are related by $\tau = \pm\kappa \cot\theta$.

2. Consider the **torus of revolution** of radii $R > r > 0$:

$$S = \left\{ (x, y, z \in \mathbb{R}^3) : \left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 = r^2 \right\}.$$

(2/20) (a) Show that S is a differentiable manifold and determine its dimension.

(2/20) (b) Check that $\mathbf{g} : (0, 2\pi) \times (0, 2\pi) \to S$ given by

$$\mathbf{g}(\theta, \varphi) = ((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\varphi, r\sin\theta)$$

is parameterization for S.

(2/20) (c) Compute $\displaystyle\int_S (x^2 + y^2) z dx \wedge dy - \left(\frac{x^3}{3} + xy^2 \right) dy \wedge dz.$

(2/20) (d) Show that the first fundamental form corresponding to the parameterization $\mathbf{g}$ is

$$\mathbf{I} = r^2 d\theta^2 + (R + r\cos\theta)^2 d\varphi^2.$$

(2/20) (e) Compute the Gauss curvature of the torus.

(2/20) (f) Is the torus a minimal surface? Why or why not?

(2/20) (g) Write the differential equations for the geodesics of the torus.

(2/20) (h) Compute the geodesic curvature of the curve $\mathbf{c}(t) = \mathbf{g}(\theta_0, t)$ by integrating the Gauss curvature on the domain $\Delta(\theta_0, \varphi_0) = \mathbf{g}((0, \theta_0) \times (0, \varphi_0))$, where $\theta_0, \varphi_0 \in (0, 2\pi)$.

(2/20) **3.** Let $S \subset \mathbb{R}^3$ be a surface whose normal unit vector makes a constant angle θ with a fixed nonzero vector $\mathbf{v}$. Prove that S is flat.