Diagrammatic categorification of the extended affine Hecke and the affine q-Schur algebras

Marco Mackaay
(joint with A.-L. Thiel)

CAMGSD and University of the Algarve, Portugal

March 27-28, 2014

Aim and open questions

Aim Explain the diagrammatic categorification of the extended affine Hecke algebra and the affine q-Schur algebra, extending work by Elias-Khovanov, Elias-Williamson, Härterich, Khovanov-Lauda, Soergel, Williamson,...

Aim and open questions

Aim Explain the diagrammatic categorification of the extended affine Hecke algebra and the affine q-Schur algebra, extending work by Elias-Khovanov, Elias-Williamson, Härterich, Khovanov-Lauda, Soergel, Williamson,...

Open questions

- Precise relation with Williamson's singular bimodules (i.e. details, such as faithfulness etc.)?

Aim and open questions

Aim Explain the diagrammatic categorification of the extended affine Hecke algebra and the affine q-Schur algebra, extending work by Elias-Khovanov, Elias-Williamson, Härterich, Khovanov-Lauda, Soergel, Williamson,...

Open questions

- Precise relation with Williamson's singular bimodules (i.e. details, such as faithfulness etc.)?
- Relation with work by Lusztig and Ginzburg-Vasserot (perverse sheaves)?

Aim and open questions

Aim Explain the diagrammatic categorification of the extended affine Hecke algebra and the affine q-Schur algebra, extending work by Elias-Khovanov, Elias-Williamson, Härterich, Khovanov-Lauda, Soergel, Williamson,...

Open questions

- Precise relation with Williamson's singular bimodules (i.e. details, such as faithfulness etc.)?
- Relation with work by Lusztig and Ginzburg-Vasserot (perverse sheaves)?
- Relation with Webster's weighted KLR algebras?

Aim and open questions

Aim Explain the diagrammatic categorification of the extended affine Hecke algebra and the affine q-Schur algebra, extending work by Elias-Khovanov, Elias-Williamson, Härterich, Khovanov-Lauda, Soergel, Williamson,...

Open questions

- Precise relation with Williamson's singular bimodules (i.e. details, such as faithfulness etc.)?
- Relation with work by Lusztig and Ginzburg-Vasserot (perverse sheaves)?
- Relation with Webster's weighted KLR algebras?
- Applications for knots/links on tori?

Aim and open questions

Aim Explain the diagrammatic categorification of the extended affine Hecke algebra and the affine q-Schur algebra, extending work by Elias-Khovanov, Elias-Williamson, Härterich, Khovanov-Lauda, Soergel, Williamson,...

Open questions

- Precise relation with Williamson's singular bimodules (i.e. details, such as faithfulness etc.)?
- Relation with work by Lusztig and Ginzburg-Vasserot (perverse sheaves)?
- Relation with Webster's weighted KLR algebras?
- Applications for knots/links on tori?
- Generalizations for elliptic Hecke algebras?

Aim and open questions

Aim Explain the diagrammatic categorification of the extended affine Hecke algebra and the affine q-Schur algebra, extending work by Elias-Khovanov, Elias-Williamson, Härterich, Khovanov-Lauda, Soergel, Williamson,...

Open questions

- Precise relation with Williamson's singular bimodules (i.e. details, such as faithfulness etc.)?
- Relation with work by Lusztig and Ginzburg-Vasserot (perverse sheaves)?
- Relation with Webster's weighted KLR algebras?
- Applications for knots/links on tori?
- Generalizations for elliptic Hecke algebras?
- Categorification of evaluation modules and their tensor products?

Plan

1. Extended affine Weyl group and extended affine Hecke algebra.

Plan

1. Extended affine Weyl group and extended affine Hecke algebra.
2. Extended affine Soergel bimodules.

Plan

1. Extended affine Weyl group and extended affine Hecke algebra.
2. Extended affine Soergel bimodules.
3. Extended affine Elias-Khovanov-Williamson diagrammatical calculus.

Plan

1. Extended affine Weyl group and extended affine Hecke algebra.
2. Extended affine Soergel bimodules.
3. Extended affine Elias-Khovanov-Williamson diagrammatical calculus.
4. Extended affine quantum general linear algebra and affine tensor space.

Plan

1. Extended affine Weyl group and extended affine Hecke algebra.
2. Extended affine Soergel bimodules.
3. Extended affine Elias-Khovanov-Williamson diagrammatical calculus.
4. Extended affine quantum general linear algebra and affine tensor space.
5. Affine Schur-Weyl duality and affine q-Schur algebra.

Plan

1. Extended affine Weyl group and extended affine Hecke algebra.
2. Extended affine Soergel bimodules.
3. Extended affine Elias-Khovanov-Williamson diagrammatical calculus.
4. Extended affine quantum general linear algebra and affine tensor space.
5. Affine Schur-Weyl duality and affine q-Schur algebra.
6. y-Deformed and extended affine Khovanov-Lauda calculus.

The extended affine Weyl group

Definition

The *extended affine Weyl group* $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is generated by

$$\sigma_1, \ldots, \sigma_r, \rho,$$

subject to the relations

$$
\begin{aligned}
\sigma_i^2 &= 1 && \text{for } i = 1, \ldots, r \\
\sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for distant } i, j = 1, \ldots, r \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, \ldots, r \\
\rho \sigma_i \rho^{-1} &= \sigma_{i+1} && \text{for } i = 1, \ldots, r
\end{aligned}
$$

The indices are understood modulo r. We say that i and j are *distant* if $j \not\equiv i \pm 1 \mod r$.

The (non-extended) affine Weyl group

Definition

The (non-extended) affine Weyl group $\mathcal{W}_{\widehat{A}_{r-1}} \subset \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is the subgroup generated by the σ_i.

Lemma

Any $w \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ can be written as

$$w = \rho^k w' = \rho^k \sigma_{i_1} \cdots \sigma_{i_l}$$

where $k \in \mathbb{Z}$ is unique and $\sigma_{i_1} \cdots \sigma_{i_l}$ is a reduced expression of an element $w' \in \mathcal{W}_{\widehat{A}_{r-1}}$.

The extended affine Hecke algebra

Definition

Similarly, the *extended affine Hecke algebra* $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is the $\mathbb{Q}(q)$-algebra generated by

$$\{T_\rho, T_\rho^{-1}, T_{\sigma_i}, i = 1, \dots r\}$$

satisfying all the relations as in $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$, except that

$$T_{\sigma_i}^2 = (q^2 - 1)T_{\sigma_i} + q^2 \quad \text{for all } i = 1, \dots, r$$

with q being a formal parameter.

A $\mathbb{Q}(q)$–basis of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is given by the set $\{T_w, w \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}\}$, with

$$T_w = T_\rho^k T_{w'} = T_\rho^k T_{\sigma_{i_1}} \cdots T_{\sigma_{i_l}}.$$

Kazhdan-Lusztig generators

Alternatively, $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is generated by

$$\{T_\rho, T_{\rho^{-1}}, b_i, i = 1, \ldots, r\}$$

where the $b_i := C'_{\sigma_i} = q^{-1}(1 + T_{\sigma_i})$ are the *Kazhdan-Lusztig generators*.

Lemma

We have:

$$\begin{aligned} b_i^2 &= (q + q^{-1})b_i && \text{for } i = 1, \ldots, r \\ b_i b_j &= b_j b_i && \text{for distant } i, j = 1, \ldots, r \\ b_i b_{i+1} b_i + b_{i+1} &= b_{i+1} b_i b_{i+1} + b_i && \text{for } i = 1, \ldots, r \\ T_\rho b_i T_\rho^{-1} &= b_{i+1} && \text{for } i = 1, \ldots, r. \end{aligned}$$

Kazhdan-Lusztig basis

Theorem (Grojnowski-Haiman)

$\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ *has the following* Kazhdan-Lusztig basis

$$\{T_\rho^k C'_w, k \in \mathbb{Z} \text{ and } w \in \mathcal{W}_{\widehat{A}_{r-1}}\},$$

with the usual positive integrality property.

Extended reflection faithful representation

Definition

$\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ acts faithfully on $R = \mathbb{Q}[y][x_1, \ldots, x_r]$:

$$\rho(x_i) = \begin{cases} x_{i+1} & \text{for } i = 1, \ldots, r-1 \\ x_1 - y & \text{for } i = r \end{cases}$$

$$\rho^{-1}(x_i) = \begin{cases} x_{i-1} & \text{for } i = 2, \ldots, r \\ x_r + y & \text{for } i = 1 \end{cases}$$

$$\sigma_j(x_i) = \begin{cases} x_{j+1} & \text{for } i = j \\ x_j & \text{for } i = j+1 \\ x_i & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \ldots, r-1$$

$$\sigma_r(x_i) = \begin{cases} x_r + y & \text{for } i = 1 \\ x_1 - y & \text{for } i = r \\ x_i & \text{otherwise} \end{cases}$$

Extended affine Soergel bimodules

Definition

For any $i = 1, \ldots, r$, define the graded bimodule

$$B_i := R \otimes_{R^{\sigma_i}} R$$

with $\deg(y) = \deg(x_k) = 2$ and

$$\begin{aligned} R^{\sigma_i} &:= \mathbb{Q}[y][x_1, \ldots, x_i + x_{i+1}, x_i x_{i+1}, \ldots, x_r] \quad (i = 1, \ldots, r-1) \\ R^{\sigma_r} &:= \mathbb{Q}[y]\,[x_2, \ldots, x_{r-1}, x_r + x_1, (x_r + y/2)\,(x_1 - y/2)] \end{aligned}$$

Define also the *twisted bimodule* $B_{\rho^{\pm 1}}$. As a left R-module, we have $B_{\rho^{\pm 1}} := R$. The right R-module structure is defined by

$$x \triangleleft a := \rho(a)^{\pm 1} x$$

for all $x \in B_{\rho^{\pm 1}}$ and $a \in R$.

Härterich's categorification theorem

Definition

The *category of extended Soergel bimodules* $\mathrm{Kar}\mathcal{E}\mathcal{B}im_{\widehat{A}_{r-1}}$, is the idempotent completion of the $\mathbb{Q}$–linear graded additive monoidal category with translation generated by the bimodules above.

Let $\mathrm{Kar}\mathcal{B}im_{\widehat{A}_{r-1}}$ be the idempotent completion of the monoidal subcategory generated by the B_i.

Härterich's categorification theorem

Definition

The *category of extended Soergel bimodules* $\mathrm{Kar}\mathcal{EB}im_{\widehat{A}_{r-1}}$, is the idempotent completion of the $\mathbb{Q}$–linear graded additive monoidal category with translation generated by the bimodules above.

Let $\mathrm{Kar}\mathcal{B}im_{\widehat{A}_{r-1}}$ be the idempotent completion of the monoidal subcategory generated by the B_i.

Theorem (Härterich)

We have

$$\mathcal{H}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{B}im_{\widehat{A}_{r-1}})$$

such that

$$C'_w \mapsto [B_w\{-1\}],$$

for any $w \in \mathcal{W}_{\widehat{A}_{r-1}}$. Here B_w (e.g. B_i) is an indecomposable bimodule uniquely determined by w (e.g. σ_i).

And its extension

Lemma

For any $i = 1, \ldots, r$, *there exist* R*–bimodule isomorphisms*

$$B_\rho^{\otimes k} \cong B_{\rho^k}$$

$$B_\rho \otimes_R B_i \cong B_{i+1} \otimes_R B_\rho$$

Corollary (M.M.-Thiel)

We have

$$\widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{EB}im_{\widehat{A}_{r-1}})$$

such that

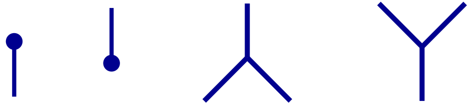
$$T_\rho^k C'_w \mapsto [B_{\rho^k} B_w\{-1\}].$$

Note that $B_\rho^{\otimes k} \ncong R$ for any $k \in \mathbb{Z}$, because the action of ρ is faithful. Putting $y = 0$ gives $B_\rho^{\otimes r} \cong R$.

The diagrammatic Soergel 2-category $\mathcal{DEB}im_{\widehat{A}_{r-1}}$

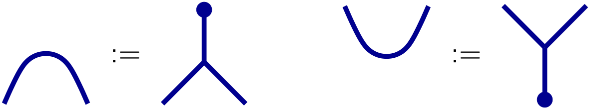
Elias-Khovanov type generators:

- involving only one color:



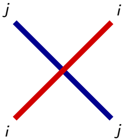
Degree	1	1	-1	-1

For simplicity, define the degree-zero morphisms:

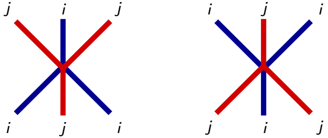


- Boxes (degree 2): $\boxed{i}$ ($i = 1, \dots, r$) and $\boxed{y}$

- The 4–valent vertex with distant colors, of degree 0:

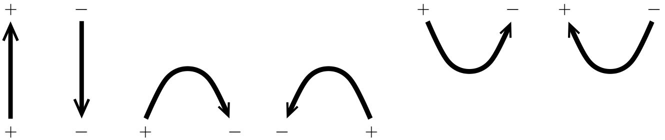


- The 6–valent vertices with adjacent colors i and j, of degree 0:

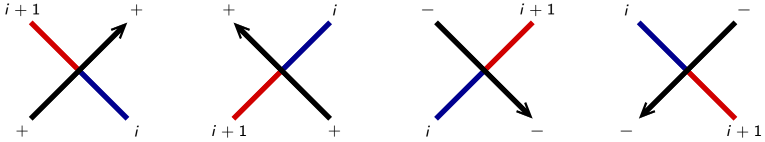


New generators:

- Generators involving only oriented strands (degree 0):

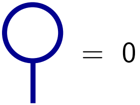


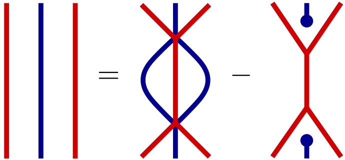
- The mixed 4–valent vertex (degree 0):

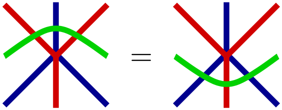


The diagrammatic Soergel 2-category $\mathcal{DEB}im_{\widehat{A}_{r-1}}$

The usual Elias-Khovanov relations, but with colors modulo r, e.g.:

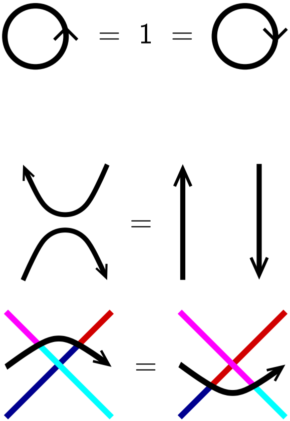






The diagrammatic Soergel 2-category $\mathcal{DEB}im_{\widehat{A}_{r-1}}$

New relations, e.g.:



The equivalence

Theorem (Elias-Williamson, M.M.-Thiel)

There exists an equivalence of graded 2-categories

$$\mathcal{DEB}im_{\widehat{A}_{r-1}} \to \mathcal{EB}im_{\widehat{A}_{r-1}}$$

for any $r \geq 3$*.*

The equivalence

Theorem (Elias-Williamson, M.M.-Thiel)

There exists an equivalence of graded 2-categories

$$\mathcal{DEB}im_{\widehat{A}_{r-1}} \to \mathcal{EB}im_{\widehat{A}_{r-1}}$$

for any $r \geq 3$.

Corollary

$$\widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{DEB}im_{\widehat{A}_{r-1}})$$

The equivalence

Theorem (Elias-Williamson, M.M.-Thiel)

There exists an equivalence of graded 2-categories

$$\mathcal{DEB}im_{\widehat{A}_{r-1}} \to \mathcal{EB}im_{\widehat{A}_{r-1}}$$

for any $r \geq 3$.

Corollary

$$\widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{DEB}im_{\widehat{A}_{r-1}})$$

Remark

The ideal generated by y is virtually nilpotent, so $K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{DEB}im_{\widehat{A}_{r-1}}/[y]) \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\mathcal{DEB}im_{\widehat{A}_{r-1}})$.

The extended affine quantum algebras

Assume $n \geq 3$ from now on.

Definition (Green)

The *extended quantum general linear algebra* $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the associative unital $\mathbb{Q}(q)$-algebra generated by $R^{\pm 1}$, $K_i^{\pm 1}$ and $E_{\pm i}$, for $i = 1, \ldots, n$, subject to the usual relations together with

$$RR^{-1} = R^{-1}R = 1 \tag{0.1}$$

$$RX_iR^{-1} = X_{i+1} \qquad \text{for } X_i \in \{E_{\pm i}, K_i^{-1}\}. \tag{0.2}$$

The extended affine quantum algebras

Assume $n \geq 3$ from now on.

Definition (Green)

The *extended quantum general linear algebra* $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the associative unital $\mathbb{Q}(q)$-algebra generated by $R^{\pm 1}$, $K_i^{\pm 1}$ and $E_{\pm i}$, for $i = 1, \ldots, n$, subject to the usual relations together with

$$RR^{-1} = R^{-1}R = 1 \tag{0.1}$$

$$RX_iR^{-1} = X_{i+1} \qquad \text{for } X_i \in \{E_{\pm i}, K_i^{-1}\}. \tag{0.2}$$

Definition

The *affine quantum general linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n) \subseteq \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is generated by $E_{\pm i}$ and $K_i^{\pm 1}$, for $i = 1, \ldots, n$.
The *affine quantum special linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n) \subseteq \mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ is generated by $E_{\pm i}$ and $K_iK_{i+1}^{-1}$, for $i = 1, \ldots, n$.

These algebras are all Hopf algebras.

The idempotented version

The degenerate level-zero $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$-weight lattice can be identified with $\mathbb{Z}^n$. Note that

$$R1_{(\lambda_1,\dots,\lambda_n)} = 1_{(\lambda_n,\lambda_1,\dots,\lambda_{n-1})}R.$$

This relation plus the usual ones give

Definition

The *idempotented extended affine quantum general linear algebra* is defined by

$$\widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n) = \bigoplus_{\lambda,\mu\in\mathbb{Z}^n} 1_\lambda \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) 1_\mu.$$

The idempotented version

Definition

Define $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \subset \widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$ as the idempotented subalgebra generated by 1_λ and $E_{\pm i}1_\lambda$, for $i = 1, \ldots, n$ and $\lambda \in \mathbb{Z}^n$.

The idempotented version

Definition

Define $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \subset \widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$ as the idempotented subalgebra generated by 1_λ and $E_{\pm i}1_\lambda$, for $i = 1, \ldots, n$ and $\lambda \in \mathbb{Z}^n$.

Definition

To define $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ adjoin an idempotent 1_μ for each $\mu \in \mathbb{Z}^{n-1}$.

The idempotented version

Definition

Define $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \subset \widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$ as the idempotented subalgebra generated by 1_λ and $E_{\pm i}1_\lambda$, for $i = 1, \ldots, n$ and $\lambda \in \mathbb{Z}^n$.

Definition

To define $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ adjoin an idempotent 1_μ for each $\mu \in \mathbb{Z}^{n-1}$.

Remark: $\mathfrak{gl}_n$-weights versus $\mathfrak{sl}_n$-weights

Recall the map $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ given by $\lambda \mapsto \overline{\lambda}$ with

$$\overline{\lambda} := (\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n).$$

Affine tensor space

Let V be the $\mathbb{Q}(q)$-vector space freely generated by $\{e_t \mid t \in \mathbb{Z}\}$.

Definition (Green)

The following defines an action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ on V

$$E_i e_{t+1} = e_t \quad \text{if } i \equiv t \mod n \tag{0.3}$$
$$E_i e_{t+1} = 0 \quad \text{if } i \not\equiv t \mod n \tag{0.4}$$
$$E_{-i} e_t = e_{t+1} \quad \text{if } i \equiv t \mod n \tag{0.5}$$
$$E_{-i} e_t = 0 \quad \text{if } i \not\equiv t \mod n \tag{0.6}$$
$$K_i^{\pm 1} e_t = q^{\pm 1} e_t \quad \text{if } i \equiv t \mod n \tag{0.7}$$
$$K_i^{\pm 1} e_t = e_t \quad \text{if } i \not\equiv t \mod n \tag{0.8}$$
$$R^{\pm 1} e_t = e_{t \pm 1} \quad \text{for all } t \in \mathbb{Z}. \tag{0.9}$$

The affine q-Schur algebra

- For any $r \in \mathbb{N}$, $V^{\otimes r}$ is a $\widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$-weight representation with weights in

$$\Lambda(n, r) = \{\lambda \in \mathbb{N}^n \colon \sum_{i=1}^{n} \lambda_i = r\}.$$

The affine q-Schur algebra

- For any $r \in \mathbb{N}$, $V^{\otimes r}$ is a $\widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$-weight representation with weights in

$$\Lambda(n,r) = \{\lambda \in \mathbb{N}^n \colon \sum_{i=1}^{n} \lambda_i = r\}.$$

- Green defined a right action of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ on $V^{\otimes r}$ which commutes with the left action of $\widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$.

Definition (Green)

Let $n, r \geq 3$. The *affine q-Schur algebra* $\widehat{\mathbf{S}}(n,r)$ is the centralizing algebra

$$\mathrm{End}_{\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}}(V^{\otimes r}).$$

Affine Schur-Weyl duality (part of)

Theorem (Green)

For $n, r \geq 3$, the image of $\psi_{n,r}\colon \widehat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n) \to End(V^{\otimes r})$ is always isomorphic to $\widehat{\mathbf{S}}(n,r)$. If $n > r$, we even have

$$\psi_{n,r}(\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)) \cong \psi_{n,r}(\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)) \cong \widehat{\mathbf{S}}(n,r).$$

For $n = r$, this is no longer true.

The case $n > r \geq 3$

Theorem (Doty-Green)

For $n > r \geq 3$, $\widehat{\mathbf{S}}(n,r)$ is isomorphic to the quotient of $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ by the ideal generated by all idempotents whose weight does not belong to $\Lambda(n,n)$.

The case $n = r \geq 3$

As vector spaces, we have

$$\widehat{\mathbf{S}}(n,n) \cong \psi_{n,n}(\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)) \oplus \bigoplus_{t \neq 0} \mathbb{Q}[R^t, R^{-t}].$$

However, this is not an algebra isomorphism.

The case $n = r \geq 3$

As vector spaces, we have

$$\widehat{\mathbf{S}}(n,n) \cong \psi_{n,n}(\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)) \oplus \bigoplus_{t\neq 0} \mathbb{Q}[R^t, R^{-t}].$$

However, this is not an algebra isomorphism.

Definition (Deng-Du-Fu)

Define

$$R^{-1} := E_{+\delta}1_n + \sum_{i=1}^{n} \sum_{a_i=0} E_{i-1}^{(a_{i-1})} \cdots E_1^{(a_1)} E_n^{(a_n)} \cdots E_{i+1}^{(a_{i+1})} 1_{(a_n,a_1,\ldots,a_{n-1})}$$

and

$$R := E_{-\delta}1_n + \sum_{i=1}^{n} \sum_{a_i=0} E_{-(i-1)}^{(a_{i-1})} \cdots E_{-1}^{(a_1)} E_{-n}^{(a_n)} \cdots E_{-(i+1)}^{(a_{i+1})} 1_{(a_1,\ldots,a_n)}.$$

The second sum is over $(a_1, \ldots, a_n) \in \Lambda(n,n)$.

Theorem (Deng-Du-Fu)

$\widehat{\mathbf{S}}(n,n)$ is generated by $E_{\pm\delta}$, $E_{\pm i}$ and 1_λ, for $i = 1, \ldots, n$ and $\lambda \in \Lambda(n,n)$, with the usual relations and ($i = 1, \ldots, n$):

i) $E_{\pm\delta}1_\lambda = 1_\lambda E_{\pm\delta} = 0$ *for all* $\lambda \neq (1^n)$;

ii) $E_{\pm\delta}1_n = 1_n E_{\pm\delta}$;

iii) $E_{+\delta}E_{-\delta}1_n = E_{-\delta}E_{+\delta}1_n = 1_n$;

iv) $E_{-i}E_{+\delta}1_n = E_{i-1} \cdots E_1 E_n \cdots E_{i+1}1_n$;

v) *etc.*

Some interesting homomorphisms

Lemma (Doty-Green, Deng-Du-Fu)

There exists an injection $\sigma_{n,r} \colon \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \to \widehat{\mathbf{S}}(n,r)$ *defined by*

$$\begin{aligned} \sigma_{n,r}(b_i) &:= 1_r E_i E_{-i} 1_r = 1_r E_{-i} E_i 1_r \\ \sigma_{n,r}(b_r) &:= 1_r E_{-n} \cdots E_{-r} E_r \cdots E_n 1_r \\ \sigma_{n,r}(T_\rho) &:= 1_r E_{-n} \cdots E_{-r-1} E_{-1} \cdots E_{-r} 1_r \\ \sigma_{n,r}(T_\rho^{-1}) &:= 1_r E_r \cdots E_1 E_{r+1} \cdots E_n 1_r, \end{aligned}$$

for $i = 1, \ldots, r-1$ *and* $n > r \geq 3$*, and*

$$\begin{aligned} \sigma_{n,r}(b_i) &:= 1_r E_i E_{-i} 1_r = 1_r E_{-i} E_i 1_r \\ \sigma_{n,r}(T_\rho^{\pm 1}) &:= 1_r E_{\pm\delta} 1_r, \end{aligned}$$

for $i = 1, \ldots, r$ *and* $n = r \geq 3$.

Some interesting homomorphisms

Lemma (Deng-Du)

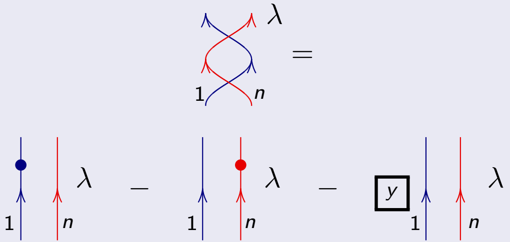
For $n \geq 3$, there exists an injection $\iota_n \colon \widehat{\mathbf{S}}(n,n) \to \widehat{\mathbf{S}}(n+1,n)$ defined by

$$\begin{array}{rcl}
1_\lambda & \mapsto & 1_{(\lambda,0)} \\
E_{\pm i}1_\lambda & \mapsto & E_{\pm i}1_{(\lambda,0)} \\
E_n 1_\lambda & \mapsto & E_n E_{n+1} 1_{(\lambda,0)} \\
E_{-n}1_\lambda & \mapsto & E_{-(n+1)}E_{-n}1_{(\lambda,0)} \\
E_{+\delta}1_n & \mapsto & E_n E_{n-1}\cdots E_1 E_{n+1}1_{(1^n,0)} \\
E_{-\delta}1_n & \mapsto & E_{-(n+1)}E_{-1}\cdots E_{-n}1_{(1^n,0)}
\end{array}$$

Categorification for $n > r \geq 3$

Definition (M.M.-Thiel)

Define $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by tensoring Khovanov and Lauda's $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ with $\mathbb{Q}[y]$ and deforming the relation



and the analogous relation with 1 and n switched and an additional minus sign.

As a consequence some bubble slide relations get deformed.

Categorification for $n > r \geq 3$

Definition (M.M-Thiel)

$\widehat{\mathcal{S}}(n,r)_{[y]}$ is the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by the ideal generated by all diagrams with regions whose label is not contained in $\Lambda(n,n)$.

Categorification for $n > r \geq 3$

Definition (M.M-Thiel)

$\widehat{\mathcal{S}}(n,r)_{[y]}$ is the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by the ideal generated by all diagrams with regions whose label is not contained in $\Lambda(n,n)$.

Theorem (M.M-Thiel)

The $\mathbb{Q}(q)$-linear algebra homomorphism

$$\gamma_{n,r} \colon \widehat{\mathbf{S}}(n,r) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

defined by

$$\gamma_{n,r}(E_{\pm i}1_\lambda) := [\mathcal{E}_{\pm i}\mathbf{1}_\lambda] \otimes 1 \quad \textit{and} \quad \gamma_{n,r}(E_{\pm\delta}1_n) := [\mathcal{E}_{\pm\delta}\mathbf{1}_n] \otimes 1$$

for any $i = 1, \ldots, n$, is a well-defined isomorphism.

Categorification for $n > r \geq 3$

Definition (M.M-Thiel)

$\widehat{\mathcal{S}}(n,r)_{[y]}$ is the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by the ideal generated by all diagrams with regions whose label is not contained in $\Lambda(n,n)$.

Theorem (M.M-Thiel)

The $\mathbb{Q}(q)$-linear algebra homomorphism

$$\gamma_{n,r} \colon \widehat{\mathbf{S}}(n,r) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

defined by

$$\gamma_{n,r}(E_{\pm i}1_\lambda) := [\mathcal{E}_{\pm i}\mathbf{1}_\lambda] \otimes 1 \quad \textit{and} \quad \gamma_{n,r}(E_{\pm\delta}1_n) := [\mathcal{E}_{\pm\delta}\mathbf{1}_n] \otimes 1$$

for any $i = 1, \ldots, n$, is a well-defined isomorphism.

Note that the ideal generated by y is virtually nilpotent, so the categorification theorem also holds for $y = 0$.

Corollary

The restriction of

$$\gamma_{n,r} : \widehat{\mathbf{S}}(n,r) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

gives rise to the composite isomorphism

$$\gamma_{n,r} \circ \sigma_{n,r} : \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \to 1_n \widehat{\mathbf{S}}(n,r) 1_n \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]}((1^r),(1^r)))$$

for any $n > r \geq 3$.

Corollary

The restriction of

$$\gamma_{n,r}\colon \widehat{\mathbf{S}}(n,r) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

gives rise to the composite isomorphism

$$\gamma_{n,r}\circ\sigma_{n,r}\colon \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \to 1_n\widehat{\mathbf{S}}(n,r)1_n \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,r)_{[y]}((1^r),(1^r)))$$

for any $n > r \geq 3$.

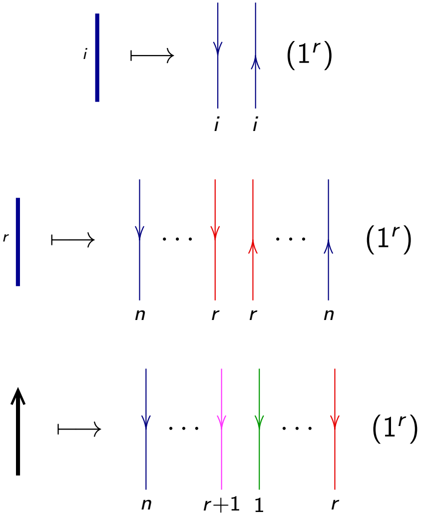
Theorem (M.M.-Thiel)

There exists a well-defined 2-functor

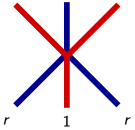
$$\Sigma_{n,r}\colon \mathcal{DEB}im^*_{\widehat{A}_{r-1}} \to \widehat{\mathcal{S}}(n,r)^*_{[y]}$$

which categorifies $\gamma_{n,r}\circ\sigma_{n,r}$, *for any* $n > r \geq 3$.

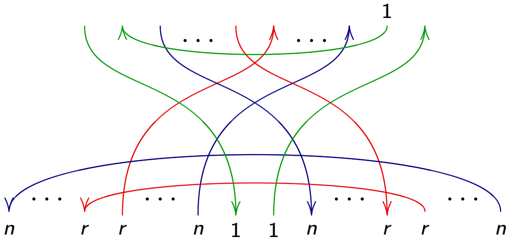
The 2-functor $\Sigma_{n,r}$, e.g.:



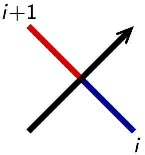
The 2-functor $\Sigma_{n,r}$, e.g.:



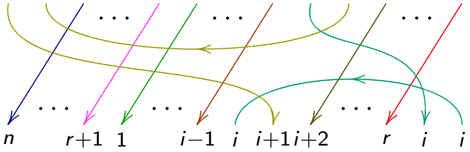
$\longmapsto$



(1^r)



$\longmapsto$



(1^r)

Relation with affine Soergel bimodules

Theorem (M.M.-Thiel)

For any $n > r \geq 3$, there exist a 2-category of extended singular affine Soergel bimodules $\mathcal{ESB}im_{\widehat{A}_{r-1}}$ and a 2-representation

$$\mathcal{F}' : \widehat{\mathcal{S}}(n,r)^*_{[y]} \to \mathcal{ESB}im^*_{\widehat{A}_{r-1}}.$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{DEB}im^*_{\widehat{A}_{r-1}} & \xrightarrow{\mathcal{F}} & \mathcal{ESB}im^*_{\widehat{A}_{r-1}} \\ & {\scriptstyle \Sigma_{n,r}} \searrow \quad \nearrow {\scriptstyle \mathcal{F}'} & \\ & \widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r)) & \end{array}$$

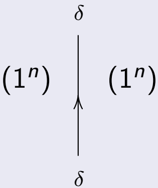
Definition (M.M.-Thiel)

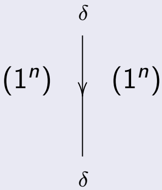
$\widehat{\mathcal{S}}(n,n)$ is the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ by the ideal generated by all diagrams with regions whose label is not contained in $\Lambda(n,n)$ (taking $y = 0$ for simplicity), together with the generating 1-morphisms

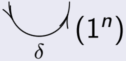
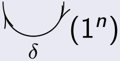
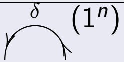
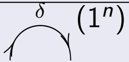
$$\mathbf{1}_n \mathcal{E}_{+\delta} \mathbf{1}_n\{t\} \quad \text{and} \quad \mathbf{1}_n \mathcal{E}_{-\delta} \mathbf{1}_n\{t\},$$

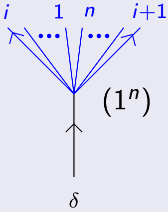
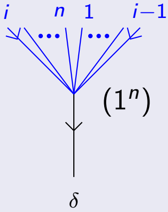
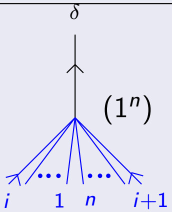
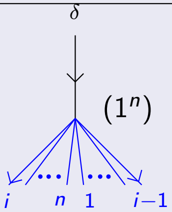
for $t \in \mathbb{Z}$, and the following generating 2-morphisms

$$1_{\mathcal{E}_{+\delta}\mathbf{1}_n\{t\}} \qquad\qquad 1_{\mathcal{E}_{-\delta}\mathbf{1}_n\{t\}}$$

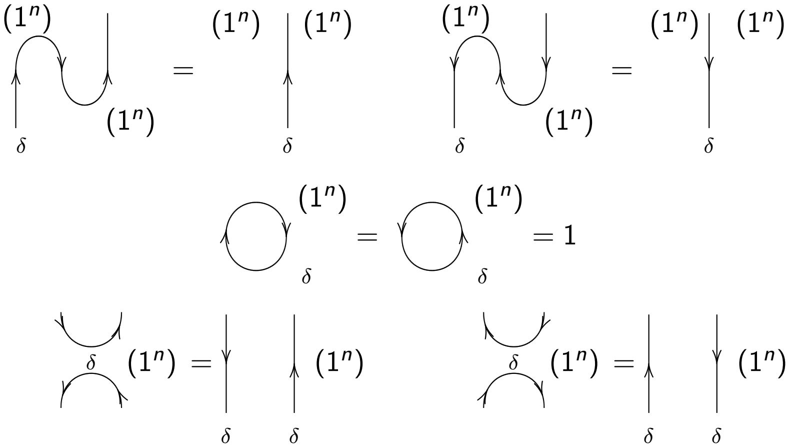




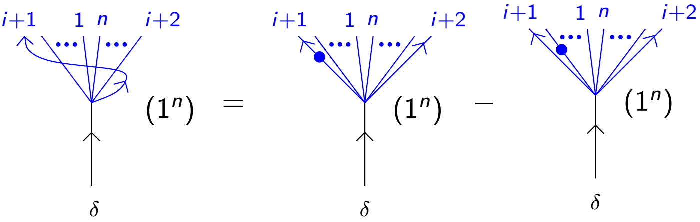
 (1^n) δ	 (1^n) δ	 δ (1^n)	 δ (1^n)
0	0	0	0

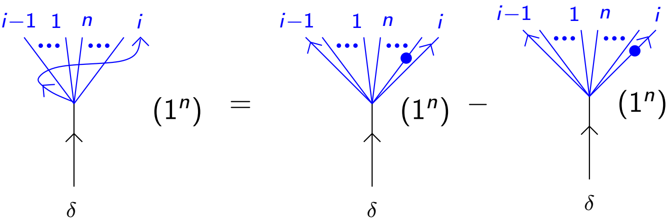
 i 1 n $i+1$ (1^n) δ	 i n 1 $i-1$ (1^n) δ	 δ (1^n) i 1 n $i+1$	 δ (1^n) i n 1 $i-1$
1	1	1	1

Extra relations

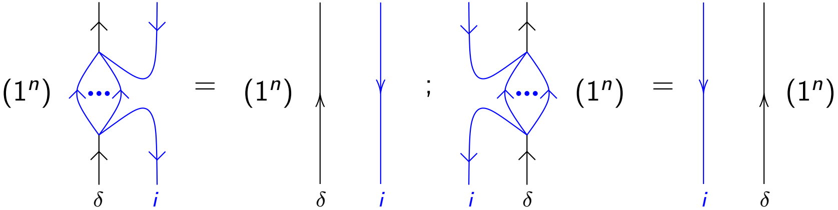


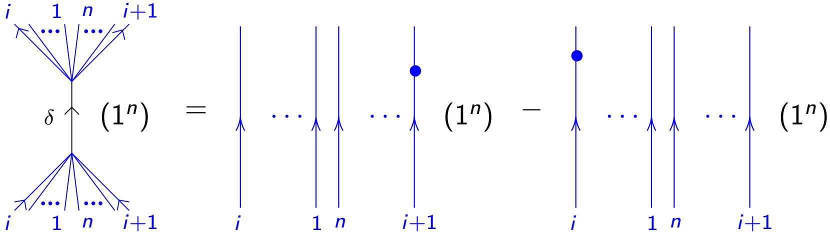
Extra relations



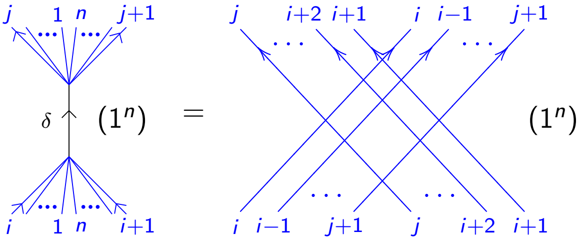


Extra relations





Extra relations



Impose cyclicity on all diagrams.

Categorification Theorem

Theorem (M.M.-Thiel)

For any $n \geq 3$, the $\mathbb{Q}(q)$-linear algebra homomorphism

$$\gamma_{n,n} \colon \widehat{\mathbf{S}}(n,n) \to K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\widehat{\mathcal{S}}(n,n))$$

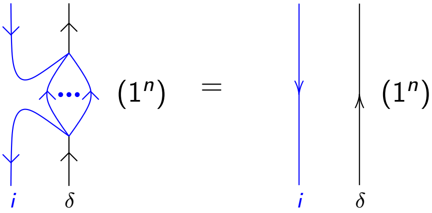
defined by

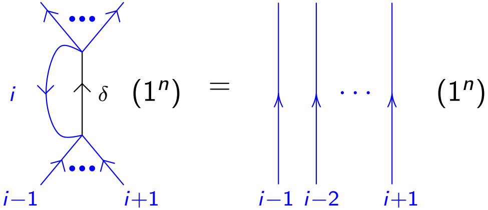
$$\gamma_{n,n}(E_{\pm i}1_\lambda) := [\mathcal{E}_{\pm i}\mathbf{1}_\lambda] \otimes 1 \quad \textit{and} \quad \gamma_{n,n}(E_{\pm\delta}1_n) := [\mathcal{E}_{\pm\delta}\mathbf{1}_n] \otimes 1$$

for any $i = 1, \ldots, n$, is a well-defined isomorphism.

Well-definedness, e.g.:

$$\mathcal{E}_{-i}\mathcal{E}_{+\delta}\mathbf{1}_n \cong \mathcal{E}_{i-1}\cdots\mathcal{E}_1\mathcal{E}_n\cdots\mathcal{E}_{i+1}\mathbf{1}_n$$

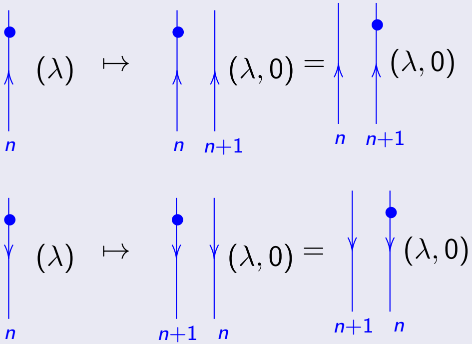




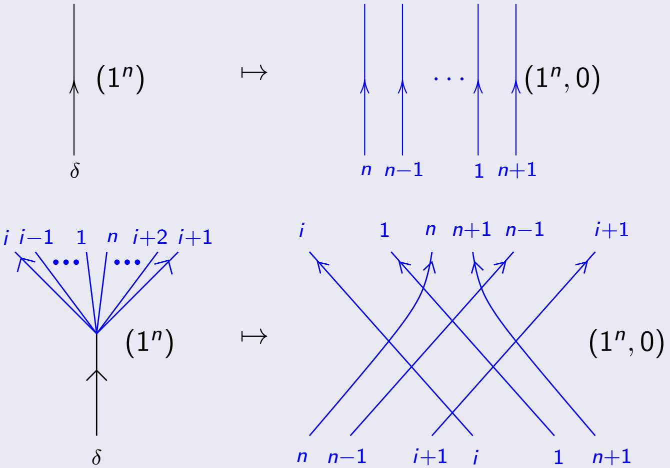
Where did we get the relations from?

Proposition (M.M.-Thiel)

The 2-functor $\mathcal{I}_n \colon \widehat{\mathcal{S}}(n,n) \to \widehat{\mathcal{S}}(n+1,n)$ *below is well-defined.*



Where did we get the relations from?



The End

THANKS!!!