Instituto Superior Técnico

Project in Mathematics

The lifting of the Eisermann invariant - some results

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Introduction

It is a central question in knot theory to decide if, given two knots, one can be smoothly transformed into the other, without cutting or creating intersections. When such a transformation is possible, we say the two knots are equivalent.

It can be very difficult to answer this question. However, over the years, many knot invariants have been discovered which can distinguish some knots. It is known, for example, that the fundamental group $\pi_1(C_K)$ of the complement of the knot C_K is a strong knot invariant, since equivalent knots have isomorphic knot groups. But it is not complete: for example it can't distinguish the trefoil and its mirror image.

Recently, Eisermann constructed an invariant $E(K)$ that is close to a complete invariant. In section 2 this invariant is explained and in section 3 we construct its lifting, following [2]. Lastly, in section 4 we present some examples that show that the lifting is indeed a stronger invariant.

1 Basic Concepts

First of all, we have to define what a knot is. Intuitively, we can construct a knot by holding a piece of string, randomly arranging it and gluing the end points together. More formally,

Definition 1. *A knot K is an embedding of the circle S^1 in $\mathbb{R}^3$.*

A knot can be represented by a knot diagram, which is simply the projection of the knot onto the plane in such a way that you can visualize if the crossings are undercrossing or overcrossings. There can be multiple knot diagrams for the same knot, which are related by the Reidemeister moves, presented in figure 1:

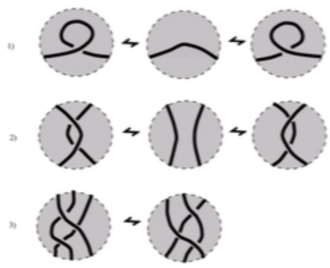


Fig. 1: Reidemeister Moves I, II and III

The Reidemeister moves correspond to the three quandle axioms, respectively, described as follows:

Definition 2. *A quandle is a set X with an operation $\triangleright$ that satisfies:*

1. $x \triangleright x = x$
2. $\forall x, y \in X \quad \exists^! z \in X : \quad y = x \triangleright z$
3. $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z), \quad \forall x, y, z \in X$

Hence, if we label the arcs of a knot diagram with elements of a quandle such that at each crossing the relation seen in figure 2 holds, then there is a one-to-one correspondence between the quandle colourings of two knot diagrams related by a Reidemeister move. Thus, in particular, the number of quandle colourings using a finite quandle is a knot invariant.

Fig. 2: Quandle colouring of a crossing on a link diagram

Let K be an oriented knot in S^3 and $C_K = S^3 \setminus n(K)$ be its complement, where $n(K)$ is a normal neighbourhood of K. The fundamental group of the complement $\pi_1(C_K)$ is known as the knot group of K and is a strong invariant. Note that the knot group is independent of the orientation of the knot.

Given a knot diagram D, each arc defines a generator of the knot group's presentation, and each crossing defines a relation. This is called the Wirtinger presentation of the knot group and is fundamental to our construction.

2 The Eisermann Invariant

Given a knot K, we can define a long knot, L_K, by cutting the knot at a point $p \in K$ and extending the end points to + and - infinity, respectively. Long knots obtained by choosing different cutting points are isotopic, and so L_K depends only on K.

We can define two special elements in the knot group: meridian and longitude. For any arc a of the knot diagram D and for any point q in a, the meridian is simply a loop around a at q, and whose direction is given by the right hand rule. For the long knot L_K take the meridian m_K to be the meridian around any point of the incoming arc, which is clearly homotopic to the meridian around any point of the the outgoing arc. The knot longitude l_K on the other hand, has linking number 0 with the knot and is mostly parallel to L_K. For a better understanding, in figure 3 [1] the longitude and meridian elements are represented for the long knot associated to the trefoil knot. For a more formal definition see [1,2].

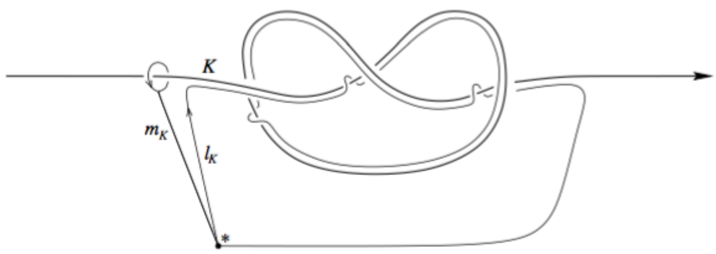


Fig. 3: Meridian and longitude for a long knot (trefoil)

Up to isomorphism, the triple $(\pi_1(C_K), l_K, m_K)$, called the peripheral system, is a complete knot invariant. The Eisermann invariant, even though it's not known to be complete, can capture some information about the peripheral system. Roughly speaking, the invariant is obtained by colouring the arcs of a knot diagram of K with elements of a finite group G associated with the meridian and the partial longitude of that arc.[2] The relations between these assignments are governed by a quandle, the Eisermann quandle, which depends on the choice of $x \in G$ associated to the meridian m_K. Finally, we register the elements in G associated to the longitude l_K.

Since the longitude l_K is an element of $\pi_1(C_K)$, it can be written in terms of the Wirtinger generators in the following way: considering L_K, we can name

[1] Image taken from [1]

[2] The partial longitude is like the knot longitude, but following the knot only as far as the arc in question

the arcs of the knot diagram $a_0, a_1, ..., a_n$, where a_0 is the first arc, that contains the base point p, and the other arcs are obtained by going along the knot in the direction of its orientation. To each arc a_i we can associate a partial longitude l_i, thus $l_K = l_n$.

Let g_{a_i} be the element of $\pi_1(C_K)$ corresponding to a_i, that is, the meridian which encircles the arc a_i. Then, at each crossing, the following relations hold:

$$X_+ : \quad l_{i+1} = l_i g_{a_i}^{-1} g_{a_j} \qquad\qquad X_- : \quad l_{i+1} = l_i g_{a_i} g_{a_j}^{-1}$$

Let G be a group. Recall that the commutator of two elements $g, h \in G$ is $[g, h] = ghg^{-1}h^{-1} \in G$. For future reference, the commutator subgroup of G, denoted G', is the subgroup of G generated by all the commutators of the group.

Let θ_i be the sign of the i-th crossing and j_i the number of the arc that splits a_i and a_{i+1}. By algebraic manipulation, for $k \in \{2, ..., n\}$, we get:

$$l_k = \prod_{i=1}^{k-1} [l_i^{-1}, g_{a_1}^{-\theta_i}][g_{a_1}^{-\theta_i}, l_{j_i}^{-1}]$$

Hence the partial longitudes l_k and the longitude l_K belong to the commutator subgroup of $\pi_1(C_K)$. Also, $[m_K, l_K] = 1$.

Given a finite group G, let $f : \pi_1(C_K) \to G$ be a group morphism. Hence, $f(l_K) \in G'$ and $[f(l_K), f(m_K)] = 1_G$. Therefore, denoting $f(m_K) = x$ and $C(x)$ the group of elements in G that commute with x, we get that

$$f(l_K) \in \Lambda = G' \cap C(x)$$

Finally, the Eisermann Invariant is given by:

$$E(K) = \sum_{\{f:\pi_1(C_K)\to G \,:\, f(m_K)=x\}} f(l_K) \quad \in \mathbb{Z}[G]$$

3 The Lifting of the Eisermann Invariant

Even though the invariant we just discussed is very strong, there is more information in C_K, the complement of K, which the Eisermann invariant can't capture, and as we'll see, the lifting can.

The main idea is to colour the arcs of the knot diagram with elements of a finite group G, as before, but also colour each crossing with an element of a finite group E. We do this in such a way that the colour of the crossing is determined by the colour of the two arcs.

Definition 3. *A crossed module of groups, $\mathcal{G} = (\partial : E \to G, \triangleright)$, is given by:*

- *a group morphism $\partial : E \to G$*
- *a left action $\triangleright$ of G on E by automorphisms (i.e. $g \triangleright (e_1 e_2) = (g \triangleright e_1)(g \triangleright e_2)$ and $g \triangleright 1 = 1$)*

such that the Peiffer equations hold:

1. $\partial(g \triangleright e) = g\partial(e)g^{-1} \quad \forall g \in G,\ \forall e \in E$
2. $\partial(e) \triangleright f = efe^{-1} \quad \forall e, f \in E$

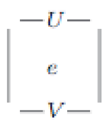
Recall the definition of left action:

Definition 4. *Let G be a group and X a set. The left group action $\triangleright$ of G on X is a function $\triangleright : G \times X \to X$ given by $(g, x) \mapsto g \triangleright x$ and satisfying:*

1. $e \triangleright x = x \quad \forall x \in X$
2. $(gh) \triangleright x = g \triangleright (h \triangleright x) \quad \forall g, h \in G,\ \forall x \in X$

Definition 5. *Given a crossed module $\mathcal{G} = (\partial : E \to G, \triangleright)$, the monoidal category $\mathcal{C}(\mathcal{G})$ is a category whose set of objects is G and whose morphisms from $U \in G$ to $V \in G$ are given by all pairs (U, e), $e \in E$ such that $\partial(e) = VU^{-1}$.*

We can see these morphisms as pointing arrows $U \xrightarrow{e} V$ and/or squares [3]:



The composition of morphisms $(U \xrightarrow{e} V) \circ (V \xrightarrow{f} W)$ is defined as $U \xrightarrow{fe} W$ and the monoidal structure $\otimes$ is expressed as

- $U \otimes V = UV$, on objects and
- $(U \xrightarrow{e} V) \otimes (U' \xrightarrow{e'} V') = UU' \xrightarrow{(V \triangleright e')e} VV'$, on morphisms.

In the square notation, the operations on morphisms are equivalent to:

[3] Figures taken from [2]

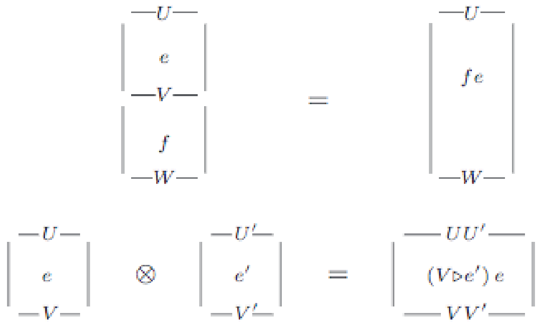


Fig. 4: Operations on morphisms of $\mathcal{C}(\mathcal{G})$ in square notation

These two operations satisfy the interchange law ie

$$((U \xrightarrow{e} V) \otimes (U' \xrightarrow{e'} V')) \circ ((V \xrightarrow{f} W) \otimes (V' \xrightarrow{f'} W')) =$$
$$= ((U \xrightarrow{e} V) \circ (V \xrightarrow{f} W)) \otimes ((U' \xrightarrow{e'} V') \circ (V' \xrightarrow{f'} W'))$$

For the following construction we use tangles, which are a generalization of braids and knots. We refer to [2] for the definition and some useful details. Analogously to knots, we can represent a tangle by a tangle diagram.

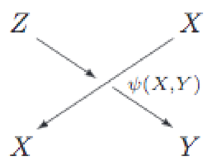
Definition 6. *Given a crossed module $\mathcal{G} = (\partial : E \to G, \triangleright)$ and an oriented tangle diagram D, a $\mathcal{G}$-colouring of D is an assignment of an element of G to each arc of D and of an element of E to each crossing of D, such that at each crossing the following relations hold:*

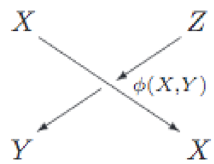
$$X_+ : \quad \partial(e) = XYX^{-1}Z^{-1} \qquad\qquad X_- : \quad \partial(e) = YXZ^{-1}X^{-1} \tag{1}$$

Thus we are assigning to each coloured crossing of D a morphism of $\mathcal{C}(\mathcal{G})$. The morphisms of $\mathcal{C}(\mathcal{G})$ associated to the elementary $\mathcal{G}$-coloured tangles can be seen in [2]. By using monoidal product horizontally and composition vertically, we can assign a morphism of $\mathcal{C}(\mathcal{G})$ to the complete $\mathcal{G}$-colouring of D, denoted F. This morphism, $e(F)$, is the evaluation of F.

In order for the element $e \in E$ associated to each crossing to be determined by the colours of the arcs, we define two functions, one for each type of crossing, which determine the E-colouring:

$X_+ : \quad \psi : G \times G \to E$ $\qquad\qquad$ $X_- : \quad \phi : G \times G \to E$





Substituing in (1), these functions then determine the remaining arc $Z \in G$:

$$X_+ : Z = \partial(\psi(X,Y))^{-1} XYX^{-1} \qquad\qquad X_- : Z = X^{-1}\partial(\phi(X,Y))^{-1} YX \quad (2)$$

Since the knot diagrams are equivalent under the Reidemeister moves, it is important that ϕ and ψ satisfy some relations, defined as follows:

Definition 7. *The pair Φ= (ψ, ϕ) is said to be an unframed Reidemeister pair if $\psi : G \times G \to E$ and $\phi : G \times G \to E$ satisfy the following three relations for each $X, Y, T \in G$:*

- *R1:* $\psi(X,X) = 1_E$
- *R2:* $\phi(X,Y)\psi(X,Z) = 1_E$
  with $Z = X^{-1}\partial(\phi(X,Y))^{-1}YX$
- *R3:* $\phi(Y,X).Y \triangleright \phi(T,Z).\phi(T,Y) = X \triangleright \phi(T,Y).\phi(T,X).T \triangleright \phi(V,W)$
  with $Z = Y^{-1}\partial(\phi(Y,X))^{-1}XY$,
  $V = T^{-1}\partial(\phi(T,Y))^{-1}YT$ *and*
  $W = T^{-1}\partial(\phi(T,X))^{-1}XT$

Example 1. *Let G' be the commutator subgroup of G and $g, h \in G'$. The pair $\Phi^x = (\psi^x, \phi^x)$ given by*

$$\phi^x(g,h) = [hx^{-1}, gx^{-1}] \qquad \psi^x(g,h) = [g,h][hg^{-1}, x] \quad (3)$$

is an unframed Reidemeister pair for the crossed module $G' \xrightarrow{id} G'$, with G' acting on itself by conjugation. Also, Φ corresponds to the quandle underlying the Eisermann invariant. These statements are proved in [2].

Hence, we can refine the definition of $\mathcal{G}$-colouring of a knot diagram D by choosing a Reidemeister pair Φ= (ψ, ϕ) and fixing the colours at each crossing as in (2). We name this colouring a Reidemeister $\mathcal{G}$-colouring of D and with it we define the state sum, as follows.

Definition 8. *Consider a crossed module $\mathcal{G} = (\partial : E \to G, \triangleright)$, G a finite group, provided with a unframed Reidemeister pair $\Phi= (\psi, \phi)$. Let D be an oriented G-enhanced tangle diagram D connecting the words w and w' in $G \sqcup G^*$. We define the state sum:*

$$I_\Phi(D) = \sum_{F \in C_\Phi(D,w,w')} e(F)$$

where $C_\Phi(D, w, w')$ is the set of Reidemeister $\mathcal{G}$-colourings of D.

Note that $I_\Phi(D) \in \mathbb{N}[Hom_{\mathcal{C}(\mathcal{G})}(e(w), e(w'))]$, where $Hom_C(x, y)$ denotes the set of morphisms $x \to y$ in the category C.

It [2] it is proved that $I_\Phi(D)$ is indeed a invariant of G-enhanced tangles if Φ is an unframed Reidemeister pair. Moreover, this invariant includes all rack and quandle cohomology invariants, as well as the Eisermann invariant.

4 New Achievements

With both the Eisermann invariant and the lifting explained, we now wish to find examples that confirm that the latter is indeed a stronger invariant. To do so, the open-source software `SageMath` and `GAP` were used.

Recall that the knot group is independent of the orientation of the knot and of the 3-dimensional space it is embedded in. However, the longitude and meridian are not, and these symmetries affect the system group as stated in [1]: if $\pi(K) = (\pi_K, m_K, l_K)$, then

$$obversion : \pi(K^\times) = (\pi_K, m_K^{-1}, l_K)$$

$$reversion : \pi(K^!) = (\pi_K, m_K^{-1}, l_K^{-1})$$

$$inversion : \pi(K^*) = (\pi_K, m_K, l_K^{-1})$$

Intuitively, obversion corresponds to mirror image and reversion to change of orientation of the knot.

We started by analyzing the trefoil knot, which is non-invertible, and then moved on to more complex knots: 5_1, which is also non-invertible, and 8_{17}, which is non-reversible.

It is known that any knot K is the closure of a braid β and in [1] Eisermann remarks that the symmetry operations on braids correspond to the symmetry operations on knots i.e. the inverse braid β^{-1} represents the inverse knot K^* and the reverse braid $\beta^!$ represents the reverse knot $K^!$.

Let β be a braid with n strings. We can assign a braid word to β which uniquely identifies it, although equivalent braids may have different braid words. The braid word associated to β is given by $i_0...i_k$, where $i_j \in \{1, ..., n-1\}$ represents a crossing with the sign of i_j between the i_j-th and $i_j + 1$-th strings, $\forall j \in \{0, ..., k\}$. Moreover, $-i_0... - i_k$ and $i_k...i_0$ are the braid words related to β^{-1} and $\beta^!$, respectively.

With this in mind, after choosing a finite group G and having the braid β corresponding to K and β_1 corresponding to K^*, in the case of the trefoil and 5_1, or $K^!$, in the case of 8_{17}, we labelled one end of the braids with elements of G and computed with (2) and (3) the labels at the other end of the braids. Since we are considering long knots, we have to impose that for each braid all the end points except the left one are equal pairwise. This way, we get F, a Reidemeister $\mathcal{G}$-colouring of the diagram. Finally, by computing the evaluation of F, by the operations on figure 4, we easily get the values of the lifting.

Consider a central extension of groups $\{0\} \to A \to E \xrightarrow{\partial} G \to \{1\}$, where G is a finite group and $\partial : E \to G$ is a surjective map, such that the kernel of ∂ is central in E. Given an arbitrary section $s : G \to E$ of ∂, that is, $\partial(s(g)) = g,\ \forall g \in G$, we get a crossed module of groups $(\partial : E \to G, \triangleright)$, with left action $g \triangleright e = s(g)\, e\, s(g)^{-1}$. This action is independent of the section s.

4.1 The Trefoil Knot: 3_1

Let K_+ and K_- be the trefoil and its mirror image, whose braid representation are as seen in figure 5.

Fig. 5: Braid diagrams corresponding to the trefoil and its mirror

In [2] some results are presented using the central extension

$$\{0\} \to \mathbb{Z}_5^* \xrightarrow{i} GL(2,5) \xrightarrow{p} PGL(2,5) \to \{1\}$$

where $GL(2,5)$ is the group of invertible two by two matrices in the field $\mathbb{Z}_5$, $\mathbb{Z}_5^*$ is the group of diagonal matrices which are multiples of the identity and $PGL(2,5)$ is the quotient group $GL(2,5)/\mathbb{Z}_5^*$.

These results state that for some $x \in PGL(2,5)$ the lifting distinguishes the knots and the Eisermannn invariant doesn't, but for other values of x both invariants distinguish them. In short, the lifting is a stronger invariant, but for this group and these knots the Eisermann invariant is already strong enough.

However, considering the central extension

$$\{0\} \to \mathbb{Z}_3^* \xrightarrow{i} GL(2,3) \xrightarrow{p} PGL(2,3) \to \{1\}$$

we get better results, as displayed in Table 1. In the first row are the values of $x \in PGL(2,3)$ for which the Eisermann invariant doesn't differentiate the trefoil from its mirror image (rows 2 and 3) and the lifting does (rows 4 and 5). For the other values of x, neither invariant distinguishes the knots. Using the same notation as in [2], if $A \in GL(2,3)$, its projection to $PGL(2,3)$ is $\tilde{A}$.

x	$\widetilde{\begin{pmatrix}1&1\\1&0\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&1\\2&1\end{pmatrix}}$	$\widetilde{\begin{pmatrix}0&1\\1&2\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&2\\2&0\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&2\\1&1\end{pmatrix}}$	$\widetilde{\begin{pmatrix}0&2\\2&2\end{pmatrix}}$
E_+	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}2&1\\1&1\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}0&2\\1&0\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}2&1\\1&1\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}2&2\\2&1\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}0&2\\1&0\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}2&2\\2&1\end{pmatrix}}$
E_-	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}2&1\\1&1\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}0&2\\1&0\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}2&1\\1&1\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}2&2\\2&1\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}0&2\\1&0\end{pmatrix}}$	$\widetilde{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + \widetilde{\begin{pmatrix}2&2\\2&1\end{pmatrix}}$
L_+	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}2&1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&2\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&2\\2&2\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}2&2\\2&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&1\\2&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&1\\1&2\end{pmatrix}$
L_-	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&2\\2&2\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&1\\2&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}2&1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&1\\1&2\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&2\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}2&2\\2&1\end{pmatrix}$

Table 1: Unlifted and lifted Eisermann invariant for the trefoil knot and its mirror image

Therefore, we have an example of a knot K and a group G with which the Eisermann invariant can't distinguish K from K^* for any $x \in G$, but for which the lifting can, for some values of x.

4.2 The Knot 5_1

Let K_+ and K_- be the 5_1 knot and its inverse, whose braid representation is pictured in figure 6.

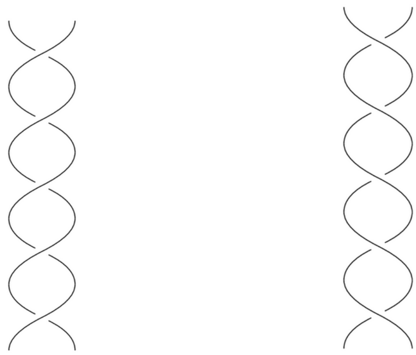


Fig. 6: Braid diagrams corresponding to the 5_1 knot and its inverse

We used the central extension $\{0\} \to \mathbb{Z}_5^* \xrightarrow{i} GL(2,5) \xrightarrow{p} PGL(2,5) \to \{1\}$. As in the case of the trefoil knot, we display some typical results in Table 2.

x	$\widetilde{\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 0 & 4 \\ 3 & 4 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 3 & 3 \end{pmatrix}}$
E_+	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 6\widetilde{\begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 6\widetilde{\begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 4\widetilde{\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 4\widetilde{\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}}$
E_-	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 6\widetilde{\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 6\widetilde{\begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 4\widetilde{\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 4\widetilde{\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}}$
L_+	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 6\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 6\begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 4 & 2 \\ 4 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$
L_-	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 6\begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 6\begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$

Table 2: Unlifted and lifted Eisermann invariant for the 5_1 knot and its mirror image

Even though for some values of $x \in PGL(2,5)$ both the Eisermann invariant and its lifting are able to distinguish these knots, as in the first and second columns, for other values the Eisermann invariant can't do so, but the lifting can, as we can see in the last two columns. Hence, once again we can conclude that the lifting is strictly stronger than the Eisermann invariant.

4.3 The Knot 8_{17}

In [1], Eisermann states that with the Mathieu group M_{11}, which has order 7920, the Eisermann invariant can distinguish this knot from its reverse. Our goal was to find a smaller group such that the lifting of the Eisermann invariant could distinguish them and the Eisermann invariant couldn't.

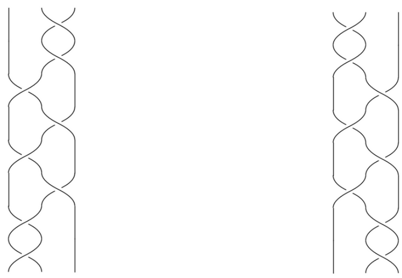


Fig. 7: Braid diagrams corresponding to the 8_{17} knot and its reverse

We did some calculations considering G as the projective special linear group $PSL(2,3)$ and as the projective general linear group $PGL(2,3)$ but with no success.

We then moved on to the central extension $\{0\} \to \mathbb{Z}_5^* \xrightarrow{i} GL(2,5) \xrightarrow{p} PGL(2,5) \to \{1\}$. Note that $PGL(2,5)$ has 120 elements and so is much smaller than the Mathieu group M_{11}. We are still searching for results like the ones we found for the previous knots, and in order to do so we restricted the choices of $x \in G$ to elements whose diagonal elements are equal since for these elements we are sure that the Eisermann invariant can't distinguish the knots [1].

Despite not being able to distinguish them, we did get some interesting results, which we display in table 3. For these values of $x \in PGL(2,5)$ the lifting shows more structure than the Eisermann invariant.

x	$\widetilde{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}}$
E_+	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 8\widetilde{\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 8\widetilde{\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}}$
E_-	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 8\widetilde{\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}}$	$\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + 8\widetilde{\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}}$
L_+	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$
L_-	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$

Table 3: Unlifted and lifted Eisermann invariant for the 8_{17} knot and its reverse

5 Conclusions

We have developed a programme for calculating the Eisermann invariant and its lifting given a braid word for the knot, a finite group G, and an element x of G. Our results include an example showing that, for $G = PGL(2, 3)$, the Eisermann invariant doesn't distinguish the trefoil from its mirror image (obverse) for any choice of x, whereas the lifted Eisermann invariant does distinguish them for six choices of x, proving that the lifted invariant is strictly stronger. This is the first example we have for which, given a group G, the lifting can distinguish two knots even though the Eisermann invariant can't.

References

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